Miscellaneous Exercise

Question 1:

Let f: **R** → **R** be defined as $f(x) = 10x + 7$. Find the function g: **R** → **R** such that

$g\ o\ f\ =\ f\ o\ g = I_R$.

Answer 1:

It is given that f: **R** → **R** is defined as $f(x) = 10x + 7$.

For one – one

Let $f(x) = f(y)$, where $x, y \in$ **R**.

$\Rightarrow 10x + 7 = 10y + 7$

$\Rightarrow x = y$

$\therefore f$ is a one – one function.

For onto

For $y \in$ **R**, let $y = 10x + 7$.

$\Rightarrow x = \dfrac{y-7}{10} \in \boldsymbol{R}$

Therefore, for any $y \in$ **R**, there exists $x = \frac{y-7}{10} \in \boldsymbol{R}$ such that

$f(x) = f\left(\frac{y-7}{10}\right) = 10\left(\frac{y-7}{10}\right) + 7 = y - 7 + 7 = y$

$\therefore f$ is onto.

Therefore, f is one – one and onto.

Thus, f is an invertible function.

Let us define g: **R** → **R** as $g(y) = \frac{y-7}{10}$

Now, we have

$$gof(x) = g\big(f(x)\big) = g(10x + 7) = \frac{(10x + 7) - 7}{10} = \frac{10x}{10} = x$$

and

$$fog(y) = f(g(y)) = f\left(\frac{y-7}{10}\right) = 10\left(\frac{y-7}{10}\right) + 7 = y - 7 + 7 = y$$

$\therefore\ g\ o\ f\ = \mathrm{I}_R\ and\ \ f\ o\ g = \mathrm{I}_{\mathbf{R}}$.

Hence, the required function g: **R** → **R** is defined as $g(y) = \frac{y-7}{10}$.

Question 2:

Let *f*: W → W be defined as *f*(*n*) = *n* − 1, if is odd and *f*(*n*) = *n* + 1, if *n* is even. Show that *f* is invertible. Find the inverse of *f*. Here, W is the set of all whole numbers.

Answer 2:

It is given that:

f: W → W is defined as $f(n) = \begin{cases} n-1, & if\ n\ is\ odd \\ n+1, & if\ n\ is\ even \end{cases}$

For one – one

Let *f*(*n*) = *f*(*m*).

It can be observed that if *n* is odd and *m* is even, then we will have *n* − 1 = *m* + 1.

⇒ *n* − *m* = 2

However, this is impossible.

Similarly, the possibility of *n* being even and *m* being odd can also be ignored under a similar argument.

∴ Both *n* and *m* must be either odd or even. Now, if both *n* and *m* are odd,

Then, we have

f(*n*) = *f*(*m*)

⇒ *n* − 1 = *m* – 1

⇒ *n* = *m*

Again, if both n and m are even,

Then, we have

$f(n) = f(m)$

$\Rightarrow n + 1 = m + 1$

$\Rightarrow n = m$

$\therefore f$ is one – one.

For onto

It is clear that any odd number $2r + 1$ in co-domain **N** is the image of $2r$ in domain **N** and any even number $2r$ in co-domain **N** is the image of $2r + 1$ in domain **N**.

$\therefore f$ is onto.

Hence, f is an invertible function.

Let us define g: W → W as $g(m) = \begin{cases} m+1, & if\ m\ is\ even \\ m-1, & if\ m\ is\ odd \end{cases}$

Now, when n is odd

$gof(n) = g(f(n)) = g(n-1) = n - 1 + 1 = n$ and

When n is even

$gof(n) = g(f(n)) = g(n+1) = n + 1 - 1 = n$

Similarly,

When m is odd

$fog(m) = f(g(m)) = f(m-1) = m - 1 + 1 = m$ and

When m is even

$fog(m) = f(g(m)) = f(m+1) = m + 1 - 1 = m$

$\therefore gof = I_W$ and $fog = I_W$

Thus, f is invertible and the inverse of f is given by $f^{-1} = g$, which is the same as f. Hence, the inverse of f is f itself.

Question 3:

If f: $\mathbf{R} \to \mathbf{R}$ is defined by $f(x) = x^2 - 3x + 2$, find $f(f(x))$.

Answer 3:

It is given that f: $\mathbf{R} \to \mathbf{R}$ is defined as $f(x) = x^2 - 3x + 2$.

$$\begin{aligned} f(f(x)) &= f(x^2 - 3x + 2) \\ &= (x^2 - 3x + 2)^2 - 3(x^2 - 3x + 2) + 2 \\ &= (x^4 + 9x^2 + 4 - 6x^3 - 12x + 4x^2) + (-3x^2 + 9x - 6) + 2 \\ &= x^4 - 6x^3 + 10x^2 - 3x \end{aligned}$$

Question 4:

Show that function f: $\mathbf{R} \to \{x \in \mathbf{R}: -1 < x < 1\}$ defined by $f(x) = \frac{x}{1+|x|}$, $x \in \mathbf{R}$ is one – one and onto function.

Answer 4:

It is given that f: $\mathbf{R} \to \{x \in \mathbf{R}: -1 < x < 1\}$ is defined as $f(x) = \frac{x}{1+|x|}$, $x \in \mathbf{R}$.

For one – one

Suppose $f(x) = f(y)$, where $x, y \in \mathbf{R}$.

$$\Rightarrow \frac{x}{1+|x|} = \frac{y}{1+|y|}$$

It can be observed that if x is positive and y is negative,

Then, we have

$$\frac{x}{1+x} = \frac{y}{1-y} \Rightarrow 2xy = x - y$$

Since, x is positive and y is negative

$x > y \Rightarrow x - y > 0$

But, $2xy$ is negative.

Then $2xy \neq x - y$

Thus, the case of x being positive and y being negative can be ruled out.

Under a similar argument, x being negative and y being positive can also be ruled out.

$\therefore$ x and y have to be either positive or negative.

When x and y are both positive, we have

$$f(x) = f(y) \Rightarrow \frac{x}{1+x} = \frac{y}{1+y} \Rightarrow x + xy = y + xy \Rightarrow x = y$$

When x and y are both negative, we have

$$f(x) = f(y) \Rightarrow \frac{x}{1-x} = \frac{y}{1-y} \Rightarrow x - xy = y - xy \Rightarrow x = y$$

$\therefore f$ is one – one.

For onto

Now, let $y \in \mathbf{R}$ such that $-1 < y < 1$.

If y is negative, then, there exists $x = \frac{y}{1+y} \in \mathbf{R}$ such that

$$f(x) = f\left(\frac{y}{1+y}\right) = \frac{\left(\frac{y}{1+y}\right)}{1 + \left|\frac{y}{1+y}\right|} = \frac{\frac{y}{1+y}}{1 + \left(\frac{-y}{1+y}\right)} = \frac{y}{1+y-y} = y$$

If y is positive, then, there exists $x = \frac{y}{1-y} \in \mathbf{R}$ such that

$$f(x) = f\left(\frac{y}{1-y}\right) = \frac{\left(\frac{y}{1-y}\right)}{1 + \left|\frac{y}{1-y}\right|} = \frac{\frac{y}{1-y}}{1 + \left(\frac{y}{1-y}\right)} = \frac{y}{1-y+y} = y$$

$\therefore f$ is onto.

Hence, f is one – one and onto.

Question 5:

Show that the function f: **R** → **R** given by $f(x) = x^3$ is injective.

Answer 5:

f: **R** → **R** is given as $f(x) = x^3$.

For one – one

Suppose $f(x) = f(y)$, where $x, y \in$ **R**.

$\Rightarrow x^3 = y^3$ .... (1)

Now, we need to show that $x = y$.

Suppose $x \neq y$, their cubes will also not be equal.

$\Rightarrow x^3 \neq y^3$

However, this will be a contradiction to (1).

$\therefore x = y$

Hence, f is injective.

Question 6:

Give examples of two functions f: **N** → **Z** and g: **Z** → **Z** such that $g \circ f$ is injective but g is not injective.

(Hint: Consider $f(x) = x$ and $g(x) = |x|$)

Answer 6:

Define f: **N** → **Z** as $f(x) = x$ and g: **Z** → **Z** as $g(x) = |x|$.

We first show that g is not injective.

It can be observed that

$g(-1) = |-1| = 1$

$g(1) = |1| = 1$

$\therefore g(-1) = g(1)$, but $-1 \neq 1$.

$\therefore g$ is not injective.

Now, g of: **N** → **Z** is defined as $gof(x) = g\big(f(x)\big) = g(x) = |x|$.

Let $x, y \in$ **N** such that g o$f(x) = g$ o$f(y)$.

$\Rightarrow |x| = |y|$

Since x and $y \in$ **N,** both are positive.

$\therefore |x| = |y| \Rightarrow x = y$

Hence, gof is injective

Question 7:

Given examples of two functions f: **N** → **N** and g: **N** → **N** such that g of is onto but f is not onto.

(Hint: Consider $f(x) = x + 1$ and $g(x) = \begin{cases} x-1, & if\ x > 1 \\ 1, & if\ x = 1 \end{cases}$

Answer 7:

Define f: **N** → **N** by $f(x) = x + 1$

and g: **N** → **N** by $g(x) = \begin{cases} x-1, & if\ x > 1 \\ 1, & if\ x = 1 \end{cases}$

We first show that g is not onto.

For this, consider element 1 in co-domain **N**. It is clear that this element is not an image of any of the elements in domain **N**.

$\therefore f$ is not onto.

Now, gof: **N** → **N** is defined by

$gof(x) = g\big(f(x)\big) = g(x+1) = x + 1 - 1 = x$ [$x \in N \Rightarrow x + 1 > 1$]

Then, it is clear that for $y \in$ **N**, there exists $x = y \in$ **N** such that $gof(x) = y$.

Hence, gof is onto.

Question 8:

Given a non-empty set X, consider P(X) which is the set of all subsets of X.

Define the relation R in P(X) as follows:

For subsets A, B in P(X), ARB if and only if $A \subset B$. Is R an equivalence relation on P(X)? Justify you answer.

Answer 8:

Since every set is a subset of itself, ARA for all $A \in$ P(X).

∴ R is reflexive.

Let AR$B \Rightarrow A \subset B$.

This cannot be implied to $B \subset A$.

For instance, if $A = \{1, 2\}$ and $B = \{1, 2, 3\}$, then it cannot be implied that B is related to A.

∴ R is not symmetric.

Further, if ARB and BRC, then $A \subset B$ and $B \subset C$.

$\Rightarrow A \subset C$

$\Rightarrow A$RC

∴ R is transitive.

Hence, R is not an equivalence relation as it is not symmetric.

Question 9:

Given a non-empty set X, consider the binary operation $*$: $P(X) \times P(X) \rightarrow P(X)$ given by $A * B = A \cap B \; \forall \; A, B$ in $P(X)$ is the power set of X. Show that X is the identity element for this operation and X is the only invertible element in $P(X)$ with respect to the operation$*$.

Answer 9:

It is given the binary operation $*$:

$P(X) \times P(X) \rightarrow P(X)$ given by $A * B = A \cap B \; \forall \; A, B$ in $P(X)$

We know that $A \cap X = A = X \cap A$ for all $A \in P(X)$

$\Rightarrow A * X = A = X * A$ for all $A \in P(X)$

Thus, X is the identity element for the given binary operation $*$.

Now, an element $A \in P(X)$ is invertible if there exists $B \in P(X)$ such that

$A * B = X = B * A$ [As X is the identity element]

or

$A \cap B = X = B \cap A$

This case is possible only when $A = X = B$.

Thus, X is the only invertible element in $P(X)$ with respect to the given operation$*$.

Hence, the given result is proved.

Question 10:

Find the number of all onto functions from the set {1, 2, 3, … , n) to itself.

Answer 10:

Onto functions from the set {1, 2, 3, … , n} to itself is simply a permutation on n symbols 1, 2, …, n.

Thus, the total number of onto maps from {1, 2, … , n} to itself is the same as the total number of permutations on n symbols 1, 2, …, n, which is n.

Question 11:

Let $S = \{a, b, c\}$ and $T = \{1, 2, 3\}$. Find F^{-1} of the following functions F from S to T, if it exists.

(i) $F = \{(a, 3), (b, 2), (c, 1)\}$ (ii) $F = \{(a, 2), (b, 1), (c, 1)\}$

Answer 11:

$S = \{a, b, c\}$, $T = \{1, 2, 3\}$

(i) F: $S \rightarrow T$ is defined as $F = \{(a, 3), (b, 2), (c, 1)\}$

$\Rightarrow F(a) = 3, F(b) = 2, F(c) = 1$

Therefore, $F^{-1}: T \rightarrow S$ is given by $F^{-1} = \{(3, a), (2, b), (1, c)\}$.

(ii) F: $S \rightarrow T$ is defined as $F = \{(a, 2), (b, 1), (c, 1)\}$

Since $F(b) = F(c) = 1$, F is not one – one.

Hence, F is not invertible i.e., F^{-1} does not exist.

Question 12:

Consider the binary operations $*: \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$ and $o: \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$ defined as $a * b = |a - b|$ and $a \text{ o } b = a$, $\forall a, b \in \mathbf{R}$. Show that * is commutative but not associative, o is associative but not commutative. Further, show that $\forall a, b, c \in \mathbf{R}$, $a*(b \text{ o } c) = (a * b) \text{ o } (a * c)$. [If it is so, we say that the operation * distributes over the operation o]. Does o distribute over *? Justify your answer.

Answer 12:

It is given that $*: \mathbf{R} \times \mathbf{R} \rightarrow$ and $o: \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$ is defined as $a * b = |a - b|$ and $a \text{ o } b = a$, $\forall a, b \in \mathbf{R}$

For $a, b \in \mathbf{R}$, we have

$a * b = |a - b|$ and $b * a = |b - a| = |-(a - b)| = |a - b|$

$\therefore a * b = b * a$

Hence, the operation * is commutative.

It can be observed that

$(1 * 2) * 3 = (|1 - 2|) * 3 = 1 * 3 = |1 - 3| = 2$

and

$1 * (2 * 3) = 1 * (|2 - 3|) = 1 * 1 = |1 - 1| = 0$

$\therefore (1 * 2) * 3 \neq 1 * (2 * 3)$ where 1, 2, 3 ∈ **R.**

Hence, the operation * is not associative.

Now, consider the operation o

It can be observed that 1 o 2 = 1 and 2 o 1 = 2.

∴ 1 o 2 ≠ 2 o 1 where 1, 2 ∈ **R.**

Hence, the operation o is not commutative.

Let $a, b, c \in$ **R**. Then, we have

$$(a \text{ o } b) \text{ o } c = a \text{ o } c = a$$

and

$$a \text{ o } (b \text{ o } c) = a \text{ o } b = a$$

∴ a o b) o c = a o (b o c), where $a, b, c \in$ **R**

Hence, the operation o is associative.

Now, let $a, b, c \in$ **R**, then we have

$a * (b \text{ o } c) = a * b = |a - b|$

$(a * b) \text{ o } (a * c) = (|a - b|)o(|a - c|) = |a - b|$

Hence, $a * (b \text{ o } c) = (a * b) \text{ o } (a * c)$.

Now,

$1\ o\ (2\ *\ 3) = 1o(|2 - 3|) = 1o1 = 1$

$(1 \text{ o } 2) * (1 \text{ o } 3) = 1 * 1 = |1 - 1| = 0$

$\therefore 1 \text{ o } (2 * 3) \neq (1 \text{ o } 2) * (1 \text{ o } 3)$ where 1, 2, 3 ∈ **R**

Hence, the operation o does not distribute over *.

Question 13:

Given a non -empty set X, let $*$: $P(X) \times P(X) \to P(X)$ be defined as $A * B = (A - B) \cup (B - A), \forall A, B \in P(X)$. Show that the empty set Φ is the identity for the operation $*$ and all the elements A of $P(X)$ are invertible with $A^{-1} = A$.

(Hint: $(A - \Phi) \cup (\Phi - A) = A$ and $(A - A) \cup (A - A) = A * A = \Phi$).

Answer 13:

It is given that $*$: $P(X) \times P(X) \to P(X)$ is defined

as $A * B = (A - B) \cup (B - A)\ \forall A, B \in P(X)$.

Let $A \in P(X)$. Then, we have

$A * \Phi = (A - \Phi) \cup (\Phi - A) = A \cup \Phi = A$

$\Phi * A = (\Phi - A) \cup (A - \Phi) = \Phi \cup A = A$

$\therefore A * \Phi = A = \Phi * A$ for all $A \in P(X)$

Thus, Φ is the identity element for the given operation$*$.

Now, an element $A \in P(X)$ will be invertible if there exists $B \in P(X)$ such that

$A * B = \Phi = B * A$. [As Φ is the identity element]

Now, we observed that

$A * A = (A - A) \cup (A - A) = \Phi \cup \Phi = \Phi$ for all $A \in P(X)$.

Hence, all the elements A of $P(X)$ are invertible with $A^{-1} = A$.

Question 14:

Define a binary operation $*$on the set {0, 1, 2, 3, 4, 5} as

$$a * b = \begin{cases} a + b, & if\ a + b < 6 \\ a + b - 6, & if\ a + b \geq 6 \end{cases}$$

Show that zero is the identity for this operation and each element $a \neq 0$ of the set is invertible with $6 - a$ being the inverse of a.

Answer 14:

Let $X = \{0, 1, 2, 3, 4, 5\}$.

The operation * on X is defined as $a * b = \begin{cases} a+b, & if\ a+b<6 \\ a+b-6, & if\ a+b \geq 6 \end{cases}$

An element $e \in X$ is the identity element for the operation *, if
$a * e = a = e * a$ for all $a \in X$

For $a \in X$, we have

$a * 0 = a + 0 = a$ $\quad [a \in X \Rightarrow a + 0 < 6]$

$0 * a = 0 + a = a$ $\quad [a \in X \Rightarrow 0 + a < 6]$

$\therefore a * 0 = a = 0 * a$ for all $a \in X$

Thus, 0 is the identity element for the given operation *.

An element $a \in X$ is invertible if there exists $b \in X$ such that $a * b = 0 = b * a$.

i.e., $\begin{cases} a+b=0=b+a, & if\ a+b<6 \\ a+b-6=0=b+a-6, & if\ a+b \geq 6 \end{cases}$

$\Rightarrow a = -b$ or $b = 6 - a$

But, $X = \{0, 1, 2, 3, 4, 5\}$ and $a, b \in X$. Then, $a \neq -b$.

$\therefore b = 6 - a$ is the inverse of a for all $a \in X$.

Hence, the inverse of an element $a \in X$, $a \neq 0$ is $6 - a$ i.e., $a^{-1} = 6 - a$.

Question 15:

Let $A = \{-1, 0, 1, 2\}$, $B = \{-4, -2, 0, 2\}$ and $f, g: A \to B$ be functions defined by $f(x) = x^2 - x, x \in$ A and $g(x) = 2\left|x - \frac{1}{2}\right| - 1,\ x \in$ A. Are f and g equal? Justify your answer. (Hint: One may note that two function $f: A \to B$ and $g: A \to$ B such that $f(a) = g(a)\ \forall\ a \in A$, are called equal functions).

Answer 15:

It is given that $A = \{-1, 0, 1, 2\}$, $B = \{-4, -2, 0, 2\}$.

Also, it is given that $f, g: A \rightarrow B$ are defined by

$f(x) = x^2 - x, x \in A$ and $g(x) = 2\left|x - \frac{1}{2}\right| - 1,\ x \in \text{A}$

It is observed that

$f(-1) = (-1)^2 - (-1) = 1 + 1 = 2$

and $g(-1) = 2\left|(-1) - \frac{1}{2}\right| - 1 = 2\left(\frac{3}{2}\right) - 1 = 3 - 1 = 2$

$\Rightarrow f(-1) = g(-1)$

$f(0) = (0)^2 - (0) = 0$

and $g(0) = 2\left|0 - \frac{1}{2}\right| - 1 = 2\left(\frac{1}{2}\right) - 1 = 1 - 1 = 0$

$\Rightarrow f(0) = g(0)$

$f(1) = (1)^2 - (1) = 1 - 1 = 0$

and $g(1) = 2\left|(1) - \frac{1}{2}\right| - 1 = 2\left(\frac{1}{2}\right) - 1 = 1 - 1 = 0$

$\Rightarrow f(1) = g(1)$

$f(2) = (2)^2 - (2) = 4 - 2 = 2$

And $g(2) = 2\left|(2) - \frac{1}{2}\right| - 1 = 2\left(\frac{3}{2}\right) - 1 = 3 - 1 = 2$

$\Rightarrow f(2) = g(2)$

$\therefore f(a) = g(a)$ for all $a \in \text{A}$

Hence, the functions f and g are equal.

Question 16:

Let $A = \{1, 2, 3\}$. Then number of relations containing (1, 2) and (1, 3) which are reflexive and symmetric but not transitive is

(A) 1 (B) 2 (C) 3 (D) 4

Answer 16:

The given set is A = {1, 2, 3}.

The smallest relation containing (1, 2) and (1, 3) which is reflexive and symmetric, but not transitive is given by:

R = {(1, 1), (2, 2), (3, 3), (1, 2), (1, 3), (2, 1), (3, 1)}

This is because relation R is reflexive as (1, 1), (2, 2), (3, 3) ∈ R.

Relation R is symmetric since (1, 2), (2, 1) ∈ R and (1, 3), (3, 1) ∈ R.

But relation R is not transitive as (3, 1), (1, 2) ∈ R, but (3, 2) ∉ R.

Now, if we add any two pairs (3, 2) and (2, 3) (or both) to relation R, then relation R will become transitive.

Hence, the total number of desired relations is one.

The correct answer is A.

Question 17:

Let $A = \{1, 2, 3\}$. Then number of equivalence relations containing (1, 2) is

(A) 1 (B) 2 (C) 3 (D) 4

Answer 17:

It is given that $A = \{1, 2, 3\}$.

The smallest equivalence relation containing (1, 2) is given by,

R_1 = {(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)}

Now, we are left with only four pairs i.e., (2, 3), (3, 2), (1, 3), and (3, 1).

If we odd any one pair [say (2, 3)] to R_1, then for symmetry we must add (3, 2).

Also, for transitivity we are required to add (1, 3) and (3, 1).

Hence, the only equivalence relation (bigger than R_1) is the universal relation.

This shows that the total number of equivalence relations containing (1, 2) is two.

The correct answer is B.

Question 18:

Let f: **R** → **R** be the Signum Function defined as $f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 1 \end{cases}$

and g: **R** → **R** be the Greatest Integer Function given by $g(x) = [x]$, where $[x]$ is greatest integer less than or equal to x. Then does fog and gof coincide in (0, 1]?

Answer 18:

It is given that,

f: **R** → **R** is defined as $f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 1 \end{cases}$

Also, g: **R** → **R** is defined as $g(x) = [x]$, where $[x]$ is the greatest integer less than or equal to x.

Now, let $x \in (0, 1]$.

Then, we have

$[x] = 1$ if $x = 1$ and $[x] = 0$ if $0 < x < 1$.

$$fog(x) = f(g(x)) = f([x]) = \begin{cases} f(1), & if\ x = 1 \\ f(0), & if\ x \in (0, 1) \end{cases}$$

$$= \begin{cases} 1, & if\ x = 1 \\ 0, & if\ x \in (0, 1) \end{cases}$$

$$gof(x) = g(f(x)) = g(1) \qquad [as\ x > 0]$$

$$= [1] = 1$$

Thus, when $x \in (0, 1)$, we have $fog(x) = 0$ and $gof(x) = 1$.

Hence, fog and gof do not coincide in $(0, 1]$.

Question 19:

Number of binary operations on the set $\{a, b\}$ are

(A) 10 (B) 16 (C) 20 (D) 8

Answer 19:

A binary operation * on $\{a, b\}$ is a function from $\{a, b\} \times \{a, b\} \to \{a, b\}$

i.e., * is a function from $\{(a, a), (a, b), (b, a), (b, b)\} \to \{a, b\}$.

Hence, the total number of binary operations on the set $\{a, b\}$ is 2^4 i.e., 16.

The correct answer is B.