

Miscellaneous Solutions

Question 1:

Prove that the determinant $\begin{vmatrix} x & \sin \theta & \cos \theta \\ -\sin \theta & -x & 1 \\ \cos \theta & 1 & x \end{vmatrix}$ is independent of θ .

Answer

$$\begin{aligned} \Delta &= \begin{vmatrix} x & \sin \theta & \cos \theta \\ -\sin \theta & -x & 1 \\ \cos \theta & 1 & x \end{vmatrix} \\ &= x(x^2 - 1) - \sin \theta(-x \sin \theta - \cos \theta) + \cos \theta(-\sin \theta + x \cos \theta) \\ &= x^3 - x + x \sin^2 \theta + \sin \theta \cos \theta - \sin \theta \cos \theta + x \cos^2 \theta \\ &= x^3 - x + x(\sin^2 \theta + \cos^2 \theta) \\ &= x^3 - x + x \\ &= x^3 \text{ (Independent of } \theta) \end{aligned}$$

Hence, Δ is independent of θ .

Question 2:

Without expanding the determinant, prove that

$$\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

Answer

$$\begin{aligned}
 \text{L.H.S.} &= \begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} \\
 &= \frac{1}{abc} \begin{vmatrix} a^2 & a^3 & abc \\ b^2 & b^3 & abc \\ c^2 & c^3 & abc \end{vmatrix} && [R_1 \rightarrow aR_1, R_2 \rightarrow bR_2, \text{ and } R_3 \rightarrow cR_3] \\
 &= \frac{1}{abc} \cdot abc \begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix} && [\text{Taking out factor } abc \text{ from } C_3] \\
 &= \begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix} \\
 &= \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} && [\text{Applying } C_1 \leftrightarrow C_3 \text{ and } C_2 \leftrightarrow C_3] \\
 &= \text{R.H.S.}
 \end{aligned}$$

Hence, the given result is proved.

Question 3:

Evaluate $\begin{vmatrix} \cos \alpha \cos \beta & \cos \alpha \sin \beta & -\sin \alpha \\ -\sin \beta & \cos \beta & 0 \\ \sin \alpha \cos \beta & \sin \alpha \sin \beta & \cos \alpha \end{vmatrix}$

Answer

$$\Delta = \begin{vmatrix} \cos \alpha \cos \beta & \cos \alpha \sin \beta & -\sin \alpha \\ -\sin \beta & \cos \beta & 0 \\ \sin \alpha \cos \beta & \sin \alpha \sin \beta & \cos \alpha \end{vmatrix}$$

Expanding along C_3 , we have:

$$\begin{aligned}
 \Delta &= -\sin \alpha (-\sin \alpha \sin^2 \beta - \cos^2 \beta \sin \alpha) + \cos \alpha (\cos \alpha \cos^2 \beta + \cos \alpha \sin^2 \beta) \\
 &= \sin^2 \alpha (\sin^2 \beta + \cos^2 \beta) + \cos^2 \alpha (\cos^2 \beta + \sin^2 \beta) \\
 &= \sin^2 \alpha (1) + \cos^2 \alpha (1) \\
 &= 1
 \end{aligned}$$

Question 4:

$$\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$$

If a , b and c are real numbers, and

Show that either $a + b + c = 0$ or $a = b = c$.

Answer

$$\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

Applying $R_1 \rightarrow R_1 + R_2 + R_3$, we have:

$$\Delta = \begin{vmatrix} 2(a+b+c) & 2(a+b+c) & 2(a+b+c) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

$$= 2(a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

Applying $C_2 \rightarrow C_2 - C_1$ and $C_3 \rightarrow C_3 - C_1$, we have:

$$\Delta = 2(a+b+c) \begin{vmatrix} 1 & 0 & 0 \\ c+a & b-c & b-a \\ a+b & c-a & c-b \end{vmatrix}$$

Expanding along R_1 , we have:

$$\begin{aligned}\Delta &= 2(a+b+c)(1)[(b-c)(c-b)-(b-a)(c-a)] \\ &= 2(a+b+c)[-b^2-c^2+2bc-bc+ba+ac-a^2] \\ &= 2(a+b+c)[ab+bc+ca-a^2-b^2-c^2]\end{aligned}$$

It is given that $\Delta = 0$.

$$(a+b+c)[ab+bc+ca-a^2-b^2-c^2] = 0$$

$$\Rightarrow \text{Either } a+b+c=0, \text{ or } ab+bc+ca-a^2-b^2-c^2=0.$$

Now,

$$ab+bc+ca-a^2-b^2-c^2=0$$

$$\Rightarrow -2ab-2bc-2ca+2a^2+2b^2+2c^2=0$$

$$\Rightarrow (a-b)^2+(b-c)^2+(c-a)^2=0$$

$$\Rightarrow (a-b)^2=(b-c)^2=(c-a)^2=0 \quad \left[(a-b)^2, (b-c)^2, (c-a)^2 \text{ are non-negative} \right]$$

$$\Rightarrow (a-b)=(b-c)=(c-a)=0$$

$$\Rightarrow a=b=c$$

Hence, if $\Delta = 0$, then either $a+b+c=0$ or $a=b=c$.

Question 5:

$$\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0, a \neq 0$$

Solve the equations

Answer

$$\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$$

Applying $R_1 \rightarrow R_1 + R_2 + R_3$, we get:

$$\begin{vmatrix} 3x+a & 3x+a & 3x+a \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$$

$$\Rightarrow (3x+a) \begin{vmatrix} 1 & 1 & 1 \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$$

Applying $C_2 \rightarrow C_2 - C_1$ and $C_3 \rightarrow C_3 - C_1$, we have:

$$(3x+a) \begin{vmatrix} 1 & 0 & 0 \\ x & a & 0 \\ x & 0 & a \end{vmatrix} = 0$$

Expanding along R_1 , we have:

$$(3x+a)[1 \times a^2] = 0$$

$$\Rightarrow a^2(3x+a) = 0$$

But $a \neq 0$.

Therefore, we have:

$$3x+a=0$$

$$\Rightarrow x = -\frac{a}{3}$$

Question 6:

$$\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$$

Prove that

Answer

$$\Delta = \begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$$

Taking out common factors a , b , and c from C_1 , C_2 , and C_3 , we have:

$$\Delta = abc \begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

Applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$, we have:

$$\Delta = abc \begin{vmatrix} a & c & a+c \\ b & b-c & -c \\ b-a & b & -a \end{vmatrix}$$

Applying $R_2 \rightarrow R_2 + R_1$, we have:

$$\Delta = abc \begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b-a & b & -a \end{vmatrix}$$

Applying $R_3 \rightarrow R_3 + R_2$, we have:

$$\Delta = abc \begin{vmatrix} a & c & a+c \\ a+b & b & a \\ 2b & 2b & 0 \end{vmatrix}$$

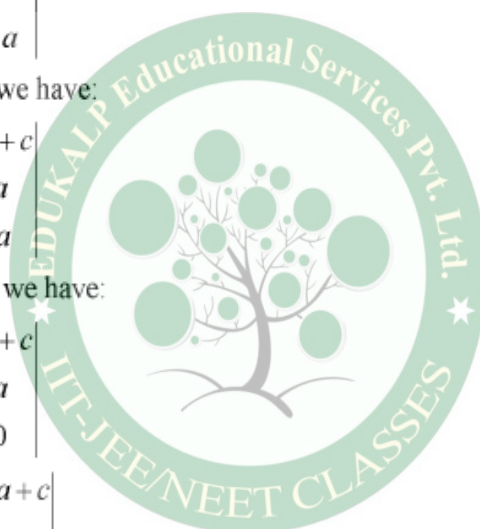
$$= 2ab^2c \begin{vmatrix} a & c & a+c \\ a+b & b & a \\ 1 & 1 & 0 \end{vmatrix}$$

Applying $C_2 \rightarrow C_2 - C_1$, we have:

$$\Delta = 2ab^2c \begin{vmatrix} a & c-a & a+c \\ a+b & -a & a \\ 1 & 0 & 0 \end{vmatrix}$$

Expanding along R_3 , we have:

$$\begin{aligned} \Delta &= 2ab^2c [a(c-a) + a(a+c)] \\ &= 2ab^2c [ac - a^2 + a^2 + ac] \\ &= 2ab^2c (2ac) \\ &= 4a^2b^2c^2 \end{aligned}$$



Hence, the given result is proved.

Question 8:

$$A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$$

Let $\quad$ verify that

(i) $[adj A]^{-1} = adj(A^{-1})$

(ii) $(A^{-1})^{-1} = A$

Answer

$$A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$$

$$\therefore |A| = 1(15 - 1) + 2(-10 - 1) + 1(-2 - 3) = 14 - 22 - 5 = -13$$

$$\text{Now, } A_{11} = 14, A_{12} = 11, A_{13} = -5$$

$$A_{21} = 11, A_{22} = 4, A_{23} = -3$$

$$A_{31} = -5, A_{32} = -3, A_{33} = -1$$

$$\therefore adj A = \begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} (adj A)$$

$$= -\frac{1}{13} \begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix} = \frac{1}{13} \begin{bmatrix} -14 & -11 & 5 \\ -11 & -4 & 3 \\ 5 & 3 & 1 \end{bmatrix}$$

(i)

$$\begin{aligned} |adj A| &= 14(-4 - 9) - 11(-11 - 15) - 5(-33 + 20) \\ &= 14(-13) - 11(-26) - 5(-13) \\ &= -182 + 286 + 65 = 169 \end{aligned}$$

We have,

$$\text{adj}(\text{adj}A) = \begin{bmatrix} -13 & 26 & -13 \\ 26 & -39 & -13 \\ -13 & -13 & -65 \end{bmatrix}$$

$$\begin{aligned} \therefore [\text{adj}A]^{-1} &= \frac{1}{|\text{adj}A|} (\text{adj}(\text{adj}A)) \\ &= \frac{1}{169} \begin{bmatrix} -13 & 26 & -13 \\ 26 & -39 & -13 \\ -13 & -13 & -65 \end{bmatrix} \\ &= \frac{1}{13} \begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & -1 \\ -1 & -1 & -5 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \text{Now, } A^{-1} &= \frac{1}{13} \begin{bmatrix} -14 & -11 & 5 \\ -11 & -4 & 3 \\ 5 & 3 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{14}{13} & -\frac{11}{13} & \frac{5}{13} \\ -\frac{11}{13} & -\frac{4}{13} & \frac{3}{13} \\ \frac{5}{13} & \frac{3}{13} & \frac{1}{13} \end{bmatrix} \\ \therefore \text{adj}(A^{-1}) &= \begin{bmatrix} -\frac{4}{169} - \frac{9}{169} & -\left(-\frac{11}{169} - \frac{15}{169}\right) & -\left(-\frac{11}{169} - \frac{15}{169}\right) \\ -\left(-\frac{11}{169} - \frac{15}{169}\right) & \frac{14}{169} - \frac{25}{169} & -\left(-\frac{42}{169} + \frac{55}{169}\right) \\ -\frac{33}{169} + \frac{20}{169} & -\left(-\frac{42}{169} + \frac{55}{169}\right) & \frac{56}{169} - \frac{121}{169} \end{bmatrix} \\ &= \frac{1}{169} \begin{bmatrix} -13 & 26 & -13 \\ 26 & -39 & -13 \\ -13 & -13 & -65 \end{bmatrix} = \frac{1}{13} \begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & -1 \\ -1 & -1 & -5 \end{bmatrix} \end{aligned}$$

Hence, $[\text{adj}A]^{-1} = \text{adj}(A^{-1})$.

(ii)

We have shown that:

$$A^{-1} = \frac{1}{13} \begin{bmatrix} -14 & -11 & 5 \\ -11 & -4 & 3 \\ 5 & 3 & 1 \end{bmatrix}$$

$$\text{And, } \text{adj} A^{-1} = \frac{1}{13} \begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & -1 \\ -1 & -1 & -5 \end{bmatrix}$$

Now,

$$|A^{-1}| = \left(\frac{1}{13}\right)^3 [-14 \times (-13) + 11 \times (-26) + 5 \times (-13)] = \left(\frac{1}{13}\right)^3 \times (-169) = -\frac{1}{13}$$

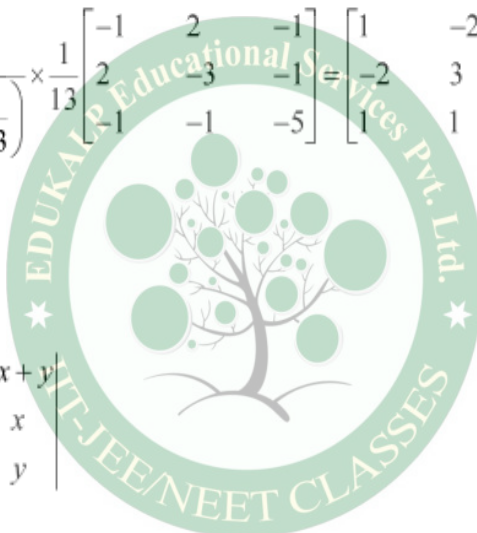
$$\therefore (A^{-1})^{-1} = \frac{\text{adj} A^{-1}}{|A^{-1}|} = \frac{1}{\left(-\frac{1}{13}\right)} \times \frac{1}{13} \begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & -1 \\ -1 & -1 & -5 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix} = A$$

$$\therefore (A^{-1})^{-1} = A$$

Question 9:

Evaluate $\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$

Answer



$$\Delta = \begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

Applying $R_1 \rightarrow R_1 + R_2 + R_3$, we have:

$$\Delta = \begin{vmatrix} 2(x+y) & 2(x+y) & 2(x+y) \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

$$= 2(x+y) \begin{vmatrix} 1 & 1 & 1 \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

Applying $C_2 \rightarrow C_2 - C_1$ and $C_3 \rightarrow C_3 - C_1$, we have:

$$\Delta = 2(x+y) \begin{vmatrix} 1 & 0 & 0 \\ y & x & x-y \\ x+y & -y & -x \end{vmatrix}$$

Expanding along R_1 , we have:

$$\begin{aligned} \Delta &= 2(x+y) [-x^2 + y(x-y)] \\ &= -2(x+y)(x^2 + y^2 - yx) \\ &= -2(x^3 + y^3) \end{aligned}$$

Question 10:

$$\text{Evaluate } \begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$$

Answer

$$\Delta = \begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$$

Applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$, we have:

$$\Delta = \begin{vmatrix} 1 & x & y \\ 0 & y & 0 \\ 0 & 0 & x \end{vmatrix}$$

Expanding along C_1 , we have:

$$\Delta = 1(xy - 0) = xy$$

Question 11:

Using properties of determinants, prove that:

$$\begin{vmatrix} \alpha & \alpha^2 & \beta + \gamma \\ \beta & \beta^2 & \gamma + \alpha \\ \gamma & \gamma^2 & \alpha + \beta \end{vmatrix} = (\beta - \gamma)(\gamma - \alpha)(\alpha - \beta)(\alpha + \beta + \gamma)$$

Answer

$$\Delta = \begin{vmatrix} \alpha & \alpha^2 & \beta + \gamma \\ \beta & \beta^2 & \gamma + \alpha \\ \gamma & \gamma^2 & \alpha + \beta \end{vmatrix}$$

Applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$, we have:

$$\begin{aligned} \Delta &= \begin{vmatrix} \alpha & \alpha^2 & \beta + \gamma \\ \beta - \alpha & \beta^2 - \alpha^2 & \alpha - \beta \\ \gamma - \alpha & \gamma^2 - \alpha^2 & \alpha - \gamma \end{vmatrix} \\ &= (\beta - \alpha)(\gamma - \alpha) \begin{vmatrix} \alpha & \alpha^2 & \beta + \gamma \\ 1 & \beta + \alpha & -1 \\ 1 & \gamma + \alpha & -1 \end{vmatrix} \end{aligned}$$

Applying $R_3 \rightarrow R_3 - R_2$, we have:

$$\Delta = (\beta - \alpha)(\gamma - \alpha) \begin{vmatrix} \alpha & \alpha^2 & \beta + \gamma \\ 1 & \beta + \alpha & -1 \\ 0 & \gamma - \beta & 0 \end{vmatrix}$$

Expanding along R_3 , we have:

$$\begin{aligned} \Delta &= (\beta - \alpha)(\gamma - \alpha) [-(\gamma - \beta)(-\alpha - \beta - \gamma)] \\ &= (\beta - \alpha)(\gamma - \alpha)(\gamma - \beta)(\alpha + \beta + \gamma) \\ &= (\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)(\alpha + \beta + \gamma) \end{aligned}$$

Hence, the given result is proved.

Question 12:

Using properties of determinants, prove that:

$$\begin{vmatrix} x & x^2 & 1 + px^3 \\ y & y^2 & 1 + py^3 \\ z & z^2 & 1 + pz^3 \end{vmatrix} = (1 + pxyz)(x - y)(y - z)(z - x)$$

Answer

$$\Delta = \begin{vmatrix} x & x^2 & 1 + px^3 \\ y & y^2 & 1 + py^3 \\ z & z^2 & 1 + pz^3 \end{vmatrix}$$

Applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$, we have:

$$\begin{aligned} \Delta &= \begin{vmatrix} x & x^2 & 1 + px^3 \\ y - x & y^2 - x^2 & p(y^3 - x^3) \\ z - x & z^2 - x^2 & p(z^3 - x^3) \end{vmatrix} \\ &= (y - x)(z - x) \begin{vmatrix} x & x^2 & 1 + px^3 \\ 1 & y + x & p(y^2 + x^2 + xy) \\ 1 & z + x & p(z^2 + x^2 + xz) \end{vmatrix} \end{aligned}$$

Applying $R_3 \rightarrow R_3 - R_2$, we have:

$$\Delta = (y-x)(z-x) \begin{vmatrix} x & x^2 & 1+px^3 \\ 1 & y+x & p(y^2+x^2+xy) \\ 0 & z-y & p(z-y)(x+y+z) \end{vmatrix}$$

$$= (y-x)(z-x)(z-y) \begin{vmatrix} x & x^2 & 1+px^3 \\ 1 & y+x & p(y^2+x^2+xy) \\ 0 & 1 & p(x+y+z) \end{vmatrix}$$

Expanding along R_3 , we have:

$$\Delta = (x-y)(y-z)(z-x) \left[(-1)(p)(xy^2+x^3+x^2y) + 1 + px^3 + p(x+y+z)(xy) \right]$$

$$= (x-y)(y-z)(z-x) \left[-pxy^2 - px^3 - px^2y + 1 + px^3 + px^2y + pxy^2 + pxyz \right]$$

$$= (x-y)(y-z)(z-x)(1+pxyz)$$

Hence, the given result is proved.

Question 13:

Using properties of determinants, prove that:

$$\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix} = 3(a+b+c)(ab+bc+ca)$$

Answer

$$\Delta = \begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix}$$

Applying $C_1 \rightarrow C_1 + C_2 + C_3$, we have:

$$\Delta = \begin{vmatrix} a+b+c & -a+b & -a+c \\ a+b+c & 3b & -b+c \\ a+b+c & -c+b & 3c \end{vmatrix}$$

$$= (a+b+c) \begin{vmatrix} 1 & -a+b & -a+c \\ 1 & 3b & -b+c \\ 1 & -c+b & 3c \end{vmatrix}$$

Applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$, we have:

$$\Delta = (a+b+c) \begin{vmatrix} 1 & -a+b & -a+c \\ 0 & 2b+a & a-b \\ 0 & a-c & 2c+a \end{vmatrix}$$

Expanding along C_1 , we have:

$$\begin{aligned} \Delta &= (a+b+c) [(2b+a)(2c+a) - (a-b)(a-c)] \\ &= (a+b+c) [4bc + 2ab + 2ac + a^2 - a^2 + ac + ba - bc] \\ &= (a+b+c) (3ab + 3bc + 3ac) \\ &= 3(a+b+c)(ab + bc + ca) \end{aligned}$$

Hence, the given result is proved.

Question 14:

Using properties of determinants, prove that:

$$\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix} = 1$$

Answer

$$\Delta = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix}$$

Applying $R_2 \rightarrow R_2 - 2R_1$ and $R_3 \rightarrow R_3 - 3R_1$, we have:

$$\Delta = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & 2+p \\ 0 & 3 & 7+3p \end{vmatrix}$$

Applying $R_3 \rightarrow R_3 - 3R_2$, we have:

$$\Delta = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & 2+p \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding along C_1 , we have:

$$\Delta = 1 \begin{vmatrix} 2+p & 1 \\ 0 & 1 \end{vmatrix} = 1(1-0) = 1$$

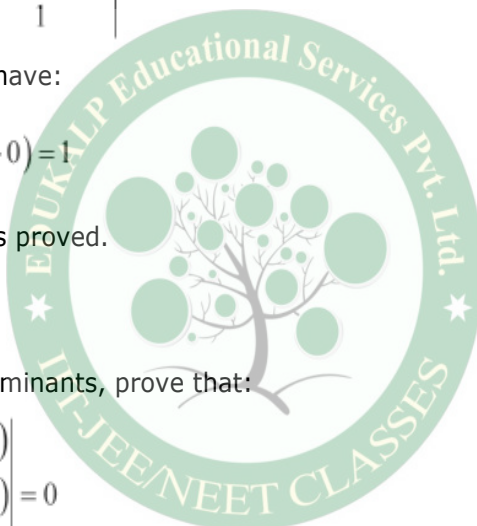
Hence, the given result is proved.

Question 15:

Using properties of determinants, prove that:

$$\begin{vmatrix} \sin \alpha & \cos \alpha & \cos(\alpha + \delta) \\ \sin \beta & \cos \beta & \cos(\beta + \delta) \\ \sin \gamma & \cos \gamma & \cos(\gamma + \delta) \end{vmatrix} = 0$$

Answer



$$\Delta = \begin{vmatrix} \sin \alpha & \cos \alpha & \cos(\alpha + \delta) \\ \sin \beta & \cos \beta & \cos(\beta + \delta) \\ \sin \gamma & \cos \gamma & \cos(\gamma + \delta) \end{vmatrix}$$

$$= \frac{1}{\sin \delta \cos \delta} \begin{vmatrix} \sin \alpha \sin \delta & \cos \alpha \cos \delta & \cos \alpha \cos \delta - \sin \alpha \sin \delta \\ \sin \beta \sin \delta & \cos \beta \cos \delta & \cos \beta \cos \delta - \sin \beta \sin \delta \\ \sin \gamma \sin \delta & \cos \gamma \cos \delta & \cos \gamma \cos \delta - \sin \gamma \sin \delta \end{vmatrix}$$

Applying $C_1 \rightarrow C_1 + C_3$, we have:

$$\Delta = \frac{1}{\sin \delta \cos \delta} \begin{vmatrix} \cos \alpha \cos \delta & \cos \alpha \cos \delta & \cos \alpha \cos \delta - \sin \alpha \sin \delta \\ \cos \beta \cos \delta & \cos \beta \cos \delta & \cos \beta \cos \delta - \sin \beta \sin \delta \\ \cos \gamma \cos \delta & \cos \gamma \cos \delta & \cos \gamma \cos \delta - \sin \gamma \sin \delta \end{vmatrix}$$

Here, two columns C_1 and C_2 are identical.

$\therefore \Delta = 0$.

Hence, the given result is proved.

Question 16:

Solve the system of the following equations

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4$$

$$\frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1$$

$$\frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2$$

Answer

$$\text{Let } \frac{1}{x} = p, \frac{1}{y} = q, \frac{1}{z} = r.$$

Then the given system of equations is as follows:

$$2p + 3q + 10r = 4$$

$$4p - 6q + 5r = 1$$

$$6p + 9q - 20r = 2$$

This system can be written in the form of $AX = B$, where

$$A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}, X = \begin{bmatrix} p \\ q \\ r \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}.$$

Now,

$$\begin{aligned} |A| &= 2(120 - 45) - 3(-80 - 30) + 10(36 + 36) \\ &= 150 + 330 + 720 \\ &= 1200 \end{aligned}$$

A

Thus, A is non-singular. Therefore, its inverse exists.

Now,

$$A_{11} = 75, A_{12} = 110, A_{13} = 72$$

$$A_{21} = 150, A_{22} = -100, A_{23} = 0$$

$$A_{31} = 75, A_{32} = 30, A_{33} = -24$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj}A$$

$$= \frac{1}{1200} \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

Now,

$$X = A^{-1}B$$

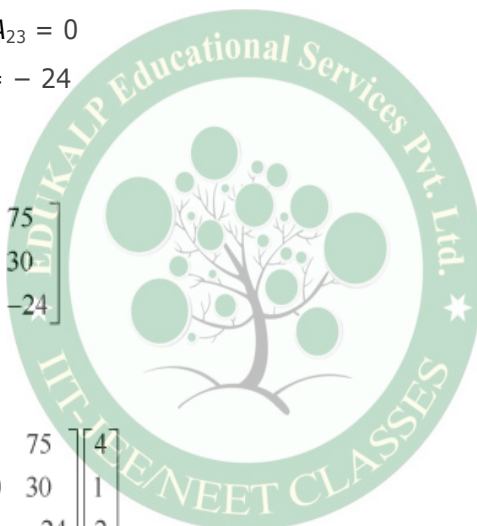
$$\Rightarrow \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \frac{1}{1200} \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{1200} \begin{bmatrix} 300 + 150 + 150 \\ 440 - 100 + 60 \\ 288 + 0 - 48 \end{bmatrix}$$

$$= \frac{1}{1200} \begin{bmatrix} 600 \\ 400 \\ 240 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{5} \end{bmatrix}$$

$$\therefore p = \frac{1}{2}, q = \frac{1}{3}, \text{ and } r = \frac{1}{5}$$

Hence, $x = 2, y = 3, \text{ and } z = 5$.



Question 17:

Choose the correct answer.

If a, b, c , are in A.P., then the determinant

$$\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$$

A. 0 B. 1 C. x D. $2x$

Answer

Answer: A

$$\Delta = \begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$$

$$= \begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+(a+c) \\ x+4 & x+5 & x+2c \end{vmatrix} \quad (2b = a+c \text{ as } a, b, \text{ and } c \text{ are in A.P.})$$

Applying $R_1 \rightarrow R_1 - R_2$ and $R_3 \rightarrow R_3 - R_2$, we have:

$$\Delta = \begin{vmatrix} -1 & -1 & a-c \\ x+3 & x+4 & x+(a+c) \\ 1 & 1 & c-a \end{vmatrix}$$

Applying $R_1 \rightarrow R_1 + R_3$, we have:

$$\Delta = \begin{vmatrix} 0 & 0 & 0 \\ x+3 & x+4 & x+a+c \\ 1 & 1 & c-a \end{vmatrix}$$

Here, all the elements of the first row (R_1) are zero.

Hence, we have $\Delta = 0$.

The correct answer is A.

Question 18:

Choose the correct answer.

$$A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$$

If x, y, z are nonzero real numbers, then the inverse of matrix

A. $\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$ **B.** $xyz \begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$

C. $\frac{1}{xyz} \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ **D.** $\frac{1}{xyz} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Answer

Answer: A

$$A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$$

$$\therefore |A| = x(yz - 0) = xyz \neq 0$$

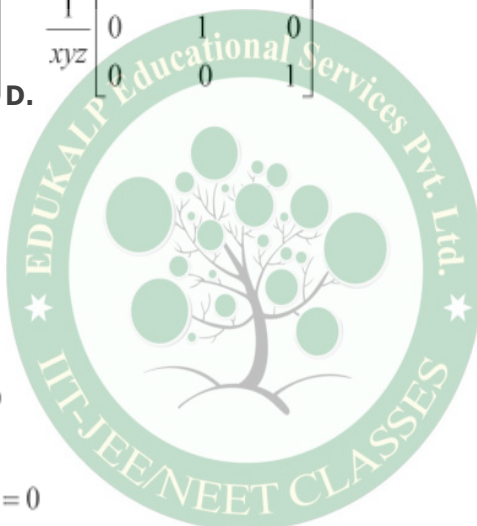
$$\text{Now, } A_{11} = yz, A_{12} = 0, A_{13} = 0$$

$$A_{21} = 0, A_{22} = xz, A_{23} = 0$$

$$A_{31} = 0, A_{32} = 0, A_{33} = xy$$

$$\therefore \text{adj}A = \begin{bmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj}A$$



$$\begin{aligned}
 &= \frac{1}{xyz} \begin{bmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{bmatrix} \\
 &= \begin{bmatrix} \frac{yz}{xyz} & 0 & 0 \\ 0 & \frac{xz}{xyz} & 0 \\ 0 & 0 & \frac{xy}{xyz} \end{bmatrix} \\
 &= \begin{bmatrix} \frac{1}{x} & 0 & 0 \\ 0 & \frac{1}{y} & 0 \\ 0 & 0 & \frac{1}{z} \end{bmatrix} = \begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}
 \end{aligned}$$

The correct answer is A.

Question 19:

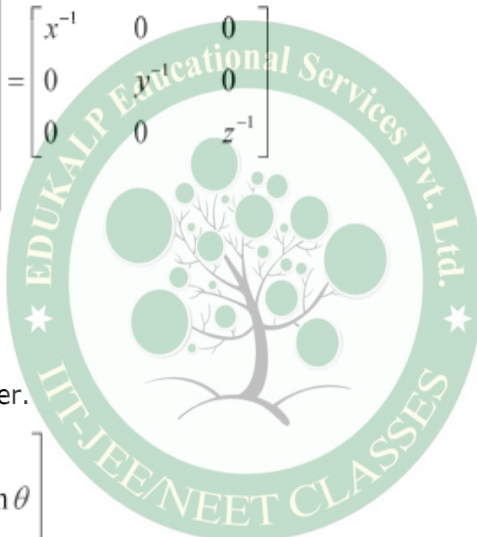
Choose the correct answer.

Let $A = \begin{bmatrix} 1 & \sin \theta & 1 \\ -\sin \theta & 1 & \sin \theta \\ -1 & -\sin \theta & 1 \end{bmatrix}$, where $0 \leq \theta \leq 2\pi$, then

A. $\text{Det}(A) = 0$

B. $\text{Det}(A) \in (2, \infty)$

C. $\text{Det}(A) \in (2, 4)$



D. $\text{Det}(A) \in [2, 4]$

Answer

Answer: D

$$A = \begin{bmatrix} 1 & \sin \theta & 1 \\ -\sin \theta & 1 & \sin \theta \\ -1 & -\sin \theta & 1 \end{bmatrix}$$

$$\begin{aligned} \therefore |A| &= 1(1 + \sin^2 \theta) - \sin \theta(-\sin \theta + \sin \theta) + 1(\sin^2 \theta + 1) \\ &= 1 + \sin^2 \theta + \sin^2 \theta + 1 \\ &= 2 + 2\sin^2 \theta \\ &= 2(1 + \sin^2 \theta) \end{aligned}$$

Now, $0 \leq \theta \leq 2\pi$

$$\Rightarrow 0 \leq \sin \theta \leq 1$$

$$\Rightarrow 0 \leq \sin^2 \theta \leq 1$$

$$\Rightarrow 1 \leq 1 + \sin^2 \theta \leq 2$$

$$\Rightarrow 2 \leq 2(1 + \sin^2 \theta) \leq 4$$

$$\therefore \text{Det}(A) \in [2, 4]$$

The correct answer is D.

