

Exercise 5.1

Question 1:

Prove that the function $f(x) = 5x - 3$ is continuous at $x = 0$, at $x = -3$ and at $x = 5$.

Answer

The given function is $f(x) = 5x - 3$

$$\text{At } x = 0, f(0) = 5 \times 0 - 3 = -3$$

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} (5x - 3) = 5 \times 0 - 3 = -3$$

$$\therefore \lim_{x \rightarrow 0} f(x) = f(0)$$

Therefore, f is continuous at $x = 0$

$$\text{At } x = -3, f(-3) = 5 \times (-3) - 3 = -18$$

$$\lim_{x \rightarrow -3} f(x) = \lim_{x \rightarrow -3} (5x - 3) = 5 \times (-3) - 3 = -18$$

$$\therefore \lim_{x \rightarrow -3} f(x) = f(-3)$$

Therefore, f is continuous at $x = -3$

$$\text{At } x = 5, f(5) = 5 \times 5 - 3 = 25 - 3 = 22$$

$$\lim_{x \rightarrow 5} f(x) = \lim_{x \rightarrow 5} (5x - 3) = 5 \times 5 - 3 = 22$$

$$\therefore \lim_{x \rightarrow 5} f(x) = f(5)$$

Therefore, f is continuous at $x = 5$

Question 2:

Examine the continuity of the function $f(x) = 2x^2 - 1$ at $x = 3$.

Answer

The given function is $f(x) = 2x^2 - 1$

$$\text{At } x = 3, f(3) = 2 \times 3^2 - 1 = 17$$

$$\lim_{x \rightarrow 3} f(x) = \lim_{x \rightarrow 3} (2x^2 - 1) = 2 \times 3^2 - 1 = 17$$

$$\therefore \lim_{x \rightarrow 3} f(x) = f(3)$$

Thus, f is continuous at $x = 3$

Question 3:

(a) The given function is

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} \frac{x^2 - 25}{x + 5} = \lim_{x \rightarrow c} \frac{(x+5)(x-5)}{x+5} = \lim_{x \rightarrow c} (x-5) = (c-5)$$

$$\text{Also, } f(c) = \frac{(c+5)(c-5)}{c+5} = (c-5) \quad (\text{as } c \neq -5)$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Hence, f is continuous at every point in the domain of f and therefore, it is a continuous function.

$$f(x) = |x-5| = \begin{cases} 5-x, & \text{if } x < 5 \\ x-5, & \text{if } x \geq 5 \end{cases}$$

(d) The given function is

This function f is defined at all points of the real line.

Let c be a point on a real line. Then, $c < 5$ or $c = 5$ or $c > 5$

Case I: $c < 5$

$$\text{Then, } f(c) = 5 - c$$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (5 - x) = 5 - c$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all real numbers less than 5.

Case II : $c = 5$

$$\text{Then, } f(c) = f(5) = (5-5) = 0$$

$$\lim_{x \rightarrow 5^-} f(x) = \lim_{x \rightarrow 5} (5 - x) = (5-5) = 0$$

$$\lim_{x \rightarrow 5^+} f(x) = \lim_{x \rightarrow 5} (x - 5) = 0$$

$$\therefore \lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x) = f(c)$$

Therefore, f is continuous at $x = 5$

Case III: $c > 5$

$$\text{Then, } f(c) = f(5) = c - 5$$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (x - 5) = c - 5$$

Question 4:

Answer

The given function is $f(x) = x^n$

It is evident that f is defined at all positive integers, n , and its value at n is n^n .

Then, $\lim_{x \rightarrow n} f(x) = \lim_{x \rightarrow n} (x^n) = n^n$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (5) = 5$$

$$\therefore \lim_{x \rightarrow 1^-} f(x) \neq \lim_{x \rightarrow 1^+} f(x)$$

Therefore, f is not continuous at $x = 1$

At $x = 2$,

f is defined at 2 and its value at 2 is 5.

$$\text{Then, } \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (5) = 5$$

Then, $f(c) = 2c - 3$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (2x - 3) = 2c - 3$$

Therefore, f is continuous at all points x , such that $x < -3$

Case II:

If $c = -3$, then $f(-3) = -(-3) + 3 = 6$

$$\lim_{x \rightarrow -3^-} f(x) = \lim_{x \rightarrow -3^-} (-x + 3) = -(-3) + 3 = 6$$

$$\lim_{x \rightarrow -3^+} f(x) = \lim_{x \rightarrow -3^+} (-2x) = -2 \times (-3) = 6$$

$$\therefore \lim_{x \rightarrow -3} f(x) = f(-3)$$

Therefore, f is continuous at $x = -3$

Case III:

If $-3 < c < 3$, then $f(c) = -2c$ and $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (-2x) = -2c$

$$f(x) = \begin{cases} \frac{|x|}{x} & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Answer

$$f(x) = \begin{cases} \frac{|x|}{x} & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

The given function f is

It is known that, $x < 0 \Rightarrow |x| = -x$ and $x > 0 \Rightarrow |x| = x$

Therefore, the given function can be rewritten as

$$f(x) = \begin{cases} \frac{|x|}{x} = \frac{-x}{x} = -1 & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ \frac{|x|}{x} = \frac{x}{x} = 1, & \text{if } x > 0 \end{cases}$$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c < 0$, then $f(c) = -1$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (-1) = -1$$

If $c > 0$, then $f(c) = 1$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (1) = 1$$

$$f(x) = \begin{cases} x+1, & \text{if } x \geq 1 \\ x^2+1, & \text{if } x < 1 \end{cases}$$

Answer

$$f(x) = \begin{cases} x+1, & \text{if } x \geq 1 \\ x^2+1, & \text{if } x < 1 \end{cases}$$

The given function f is

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c < 1$, then $f(c) = c^2 + 1$ and $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (x^2 + 1) = c^2 + 1$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x < 1$

Case II:

If $c = 1$, then $f(c) = f(1) = 1 + 1 = 2$

The left hand limit of f at $x = 1$ is,

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (x^2 + 1) = 1^2 + 1 = 2$$

The right hand limit of f at $x = 1$ is,

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (x + 1) = 1 + 1 = 2$$

$$\therefore \lim_{x \rightarrow 1} f(x) = f(1)$$

Therefore, f is continuous at $x = 1$

Case III:

If $c > 1$, then $f(c) = c + 1$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (x + 1) = c + 1$$

$$f(x) = \begin{cases} x^3 - 3, & \text{if } x \leq 2 \\ x^2 + 1, & \text{if } x > 2 \end{cases}$$

Answer

$$f(x) = \begin{cases} x^3 - 3, & \text{if } x \leq 2 \\ x^2 + 1, & \text{if } x > 2 \end{cases}$$

The given function f is

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c < 2$, then $f(c) = c^3 - 3$ and $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (x^3 - 3) = c^3 - 3$

$$f(x) = \begin{cases} x^{10} - 1, & \text{if } x \leq 1 \\ x^2, & \text{if } x > 1 \end{cases}$$

Answer

$$f(x) = \begin{cases} x^{10} - 1, & \text{if } x \leq 1 \\ x^2, & \text{if } x > 1 \end{cases}$$

The given function f is

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c < 1$, then $f(c) = c^{10} - 1$ and $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (x^{10} - 1) = c^{10} - 1$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x < 1$

Case II:

If $c = 1$, then the left hand limit of f at $x = 1$ is,

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (x^{10} - 1) = 1^{10} - 1 = 1 - 1 = 0$$

The right hand limit of f at $x = 1$ is,

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (x^2) = 1^2 = 1$$

It is observed that the left and right hand limit of f at $x = 1$ do not coincide.

Therefore, f is not continuous at $x = 1$

Case III:

If $c > 1$, then $f(c) = c^2$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (x^2) = c^2$$

$$f(x) = \begin{cases} x+5, & \text{if } x \leq 1 \\ x-5, & \text{if } x > 1 \end{cases}$$

a continuous function?

Answer

$$f(x) = \begin{cases} x+5, & \text{if } x \leq 1 \\ x-5, & \text{if } x > 1 \end{cases}$$

The given function is

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c < 1$, then $f(c) = c+5$ and $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (x+5) = c+5$

Discuss the continuity of the function f , where f is defined by

$$f(x) = \begin{cases} 3, & \text{if } 0 \leq x \leq 1 \\ 4, & \text{if } 1 < x < 3 \\ 5, & \text{if } 3 \leq x \leq 10 \end{cases}$$

Answer

$$f(x) = \begin{cases} 3, & \text{if } 0 \leq x \leq 1 \\ 4, & \text{if } 1 < x < 3 \\ 5, & \text{if } 3 \leq x \leq 10 \end{cases}$$

The given function is

The given function is defined at all points of the interval $[0, 10]$.

Let c be a point in the interval $[0, 10]$.

Case I:

If $0 \leq c < 1$, then $f(c) = 3$ and $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (3) = 3$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous in the interval $[0, 1)$.

Case II:

The left hand limit of f at $x = 3$ is,

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} (4) = 4$$

The right hand limit of f at $x = 3$ is,

$$\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} (5) = 5$$

It is observed that the left and right hand limits of f at $x = 3$ do not coincide.

Therefore, f is not continuous at $x = 3$

Case V:

If $3 < c \leq 10$, then $f(c) = 5$ and $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (5) = 5$

Case II:

If $c = 0$, then $f(c) = f(0) = 0$

The left hand limit of f at $x = 0$ is,

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (2x) = 2 \times 0 = 0$$

The right hand limit of f at $x = 0$ is,

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (0) = 0$$

$$\therefore \lim_{x \rightarrow 0} f(x) = f(0)$$

Therefore, f is continuous at $x = 0$

Case III:

If $0 < c < 1$, then $f(x) = 0$ and $\lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^-} (0) = 0$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points of the interval $(0, 1)$.

Case IV:

If $c = 1$, then $f(c) = f(1) = 0$

The left hand limit of f at $x = 1$ is,

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (0) = 0$$

The right hand limit of f at $x = 1$ is,

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (4x) = 4 \times 1 = 4$$

It is observed that the left and right hand limits of f at $x = 1$ do not coincide.

Therefore, f is not continuous at $x = 1$

Case V:

If $c < 1$, then $f(c) = 4c$ and $\lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^-} (4x) = 4c$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x < 1$

Hence, f is not continuous only at $x = 1$

Question 16:

Discuss the continuity of the function f , where f is defined by

$$f(x) = \begin{cases} -2, & \text{if } x \leq -1 \\ 2x, & \text{if } -1 < x \leq 1 \\ 2, & \text{if } x > 1 \end{cases}$$

Answer

$$f(x) = \begin{cases} -2, & \text{if } x \leq -1 \\ 2x, & \text{if } -1 < x \leq 1 \\ 2, & \text{if } x > 1 \end{cases}$$

The given function f is

The given function is defined at all points of the real line.

Let c be a point on the real line.

Case I:

If $c < -1$, then $f(c) = -2$ and $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (-2) = -2$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x < -1$

Case II:

If $c = -1$, then $f(c) = f(-1) = -2$

The left hand limit of f at $x = -1$ is,

$$\lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^-} (-2) = -2$$

The right hand limit of f at $x = -1$ is,

$$\lim_{x \rightarrow -1^+} f(x) = \lim_{x \rightarrow -1^+} (2x) = 2 \times (-1) = -2$$

$$\therefore \lim_{x \rightarrow -1} f(x) = f(-1)$$

Therefore, f is continuous at $x = -1$

Case III:

If $-1 < c < 1$, then $f(c) = 2c$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (2x) = 2c$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points of the interval $(-1, 1)$.

Case IV:

If $c = 1$, then $f(c) = f(1) = 2 \times 1 = 2$

The left hand limit of f at $x = 1$ is,

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (2x) = 2 \times 1 = 2$$

The right hand limit of f at $x = 1$ is,

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} 2 = 2$$

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^+} f(x) = f(3) \quad \dots(1)$$

Also,

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} (ax+1) = 3a+1$$

$$\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} (bx+3) = 3b+3$$

$$f(3) = 3a+1$$

Therefore, from (1), we obtain

$$3a+1 = 3b+3 = 3a+1$$

$$\Rightarrow 3a+1 = 3b+3$$

$$\Rightarrow 3a = 3b+2$$

$$\Rightarrow a = b + \frac{2}{3}$$

Therefore, the required relationship is given by,

At $x = 1$,

$$f(1) = 4x + 1 = 4 \times 1 + 1 = 5$$

$$\lim_{x \rightarrow 1} (4x + 1) = 4 \times 1 + 1 = 5$$

$$\therefore \lim_{x \rightarrow 1} f(x) = f(1)$$

Therefore, for any values of λ , f is continuous at $x = 1$

Question 19:

Show that the function defined by

It is evident that f is defined at $x = p$

$$\text{At } x = \pi, f(x) = f(\pi) = \pi^2 - \sin \pi + 5 = \pi^2 - 0 + 5 = \pi^2 + 5$$

$$\text{Consider } \lim_{x \rightarrow \pi} f(x) = \lim_{x \rightarrow \pi} (x^2 - \sin x + 5)$$

Put $x = \pi + h$

If $x \rightarrow \pi$, then it is evident that $h \rightarrow 0$

$$\begin{aligned} \therefore \lim_{x \rightarrow \pi} f(x) &= \lim_{x \rightarrow \pi} (x^2 - \sin x + 5) \\ &= \lim_{h \rightarrow 0} [(\pi + h)^2 - \sin(\pi + h) + 5] \\ &= \lim_{h \rightarrow 0} (\pi + h)^2 - \lim_{h \rightarrow 0} \sin(\pi + h) + \lim_{h \rightarrow 0} 5 \\ &= (\pi + 0)^2 - \lim_{h \rightarrow 0} [\sin \pi \cosh + \cos \pi \sinh] + 5 \\ &= \pi^2 - \lim_{h \rightarrow 0} \sin \pi \cosh - \lim_{h \rightarrow 0} \cos \pi \sinh + 5 \\ &= \pi^2 - \sin \pi \cos 0 - \cos \pi \sin 0 + 5 \\ &= \pi^2 - 0 \times 1 - (-1) \times 0 + 5 \\ &= \pi^2 + 5 \\ \therefore \lim_{x \rightarrow \pi} f(x) &= f(\pi) \end{aligned}$$

Therefore, the given function f is continuous at $x = \pi$

Question 21:

Discuss the continuity of the following functions.

(a) $f(x) = \sin x + \cos x$

(b) $f(x) = \sin x - \cos x$

(c) $f(x) = \sin x \times \cos x$

Answer

It is known that if g and h are two continuous functions, then

$$g(c) = \sin c$$

$$\begin{aligned}\lim_{x \rightarrow c} g(x) &= \lim_{x \rightarrow c} \sin x \\ &= \lim_{h \rightarrow 0} \sin(c+h) \\ &= \lim_{h \rightarrow 0} [\sin c \cos h + \cos c \sin h] \\ &= \lim_{h \rightarrow 0} (\sin c \cos h) + \lim_{h \rightarrow 0} (\cos c \sin h) \\ &= \sin c \cos 0 + \cos c \sin 0 \\ &= \sin c + 0 \\ &= \sin c\end{aligned}$$

$$\therefore \lim_{x \rightarrow c} g(x) = g(c)$$

Therefore, g is a continuous function.

Let $h(x) = \cos x$

It is evident that $h(x) = \cos x$ is defined for every real number.

Let c be a real number. Put $x = c + h$

If $x \rightarrow c$, then $h \rightarrow 0$

$$h(c) = \cos c$$

$$\begin{aligned}\lim_{x \rightarrow c} h(x) &= \lim_{x \rightarrow c} \cos x \\ &= \lim_{h \rightarrow 0} \cos(c+h) \\ &= \lim_{h \rightarrow 0} [\cos c \cos h - \sin c \sin h] \\ &= \lim_{h \rightarrow 0} \cos c \cos h - \lim_{h \rightarrow 0} \sin c \sin h \\ &= \cos c \cos 0 - \sin c \sin 0 \\ &= \cos c \times 1 - \sin c \times 0 \\ &= \cos c\end{aligned}$$

$$\therefore \lim_{x \rightarrow c} h(x) = h(c)$$

Therefore, h is a continuous function.

Therefore, it can be concluded that

(a) $f(x) = g(x) + h(x) = \sin x + \cos x$ is a continuous function

(b) $f(x) = g(x) - h(x) = \sin x - \cos x$ is a continuous function

(c) $f(x) = g(x) \times h(x) = \sin x \times \cos x$ is a continuous function

Question 22:

Discuss the continuity of the cosine, cosecant, secant and cotangent functions,

Answer

It is known that if g and h are two continuous functions, then

(i) $\frac{h(x)}{g(x)}$, $g(x) \neq 0$ is continuous

(ii) $\frac{1}{g(x)}$, $g(x) \neq 0$ is continuous

(iii) $\frac{1}{h(x)}$, $h(x) \neq 0$ is continuous

It has to be proved first that $g(x) = \sin x$ and $h(x) = \cos x$ are continuous functions.

Let $g(x) = \sin x$

It is evident that $g(x) = \sin x$ is defined for every real number.

Let c be a real number. Put $x = c + h$

If x

$$\begin{aligned}
 \lim_{x \rightarrow c} h(x) &= \lim_{x \rightarrow c} \cos x \\
 &= \lim_{h \rightarrow 0} \cos(c+h) \\
 &= \lim_{h \rightarrow 0} [\cos c \cos h - \sin c \sin h] \\
 &= \lim_{h \rightarrow 0} \cos c \cos h - \lim_{h \rightarrow 0} \sin c \sin h \\
 &= \cos c \cos 0 - \sin c \sin 0 \\
 &= \cos c \times 1 - \sin c \times 0 \\
 &= \cos c
 \end{aligned}$$

$$\therefore \lim_{x \rightarrow c} h(x) = h(c)$$

Therefore, $h(x) = \cos x$ is continuous function.

It can be concluded that,

$$\operatorname{cosec} x = \frac{1}{\sin x}, \sin x \neq 0 \text{ is continuous}$$

$$\Rightarrow \operatorname{cosec} x, x \neq n\pi \ (n \in \mathbb{Z}) \text{ is continuous}$$

Therefore, cosecant is continuous except at $x = n\pi, n \in \mathbb{Z}$

$$\sec x = \frac{1}{\cos x}, \cos x \neq 0 \text{ is continuous}$$

$$\Rightarrow \sec x, x \neq (2n+1)\frac{\pi}{2} \ (n \in \mathbb{Z}) \text{ is continuous}$$

Therefore, secant is continuous except at $x = (2n+1)\frac{\pi}{2} \ (n \in \mathbb{Z})$

$$\cot x = \frac{\cos x}{\sin x}, \sin x \neq 0 \text{ is continuous}$$

$$\Rightarrow \cot x, x \neq n\pi \ (n \in \mathbb{Z}) \text{ is continuous}$$

Therefore, cotangent is continuous except at $x = n\pi, n \in \mathbb{Z}$

Question 23:

Find the points of discontinuity of f , where

$$f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x < 0 \\ x+1, & \text{if } x \geq 0 \end{cases}$$

Answer

$$f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x < 0 \\ x+1, & \text{if } x \geq 0 \end{cases}$$

The given function f is

It is evident that f is defined at all points of the real line.

Let c be a real number.

Case I:

$$\text{If } c < 0, \text{ then } f(c) = \frac{\sin c}{c} \text{ and } \lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} \left(\frac{\sin x}{x} \right) = \frac{\sin c}{c}$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x < 0$

Case II:

$$\text{If } c > 0, \text{ then } f(c) = c+1 \text{ and } \lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (x+1) = c+1$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x > 0$

Case III:

$$\text{If } c = 0, \text{ then } f(c) = f(0) = 0+1=1$$

The left hand limit of f at $x = 0$ is,

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{\sin x}{x} = 1$$

The right hand limit of f at $x = 0$ is,

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (x+1) = 1$$

$$\therefore \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0^+} f(x) = f(0)$$

Therefore, f is continuous at $x = 0$

From the above observations, it can be concluded that f is continuous at all points of the real line.

Thus, f has no point of discontinuity.

Question 24:

Determine if f defined by

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

is a continuous function?

Answer

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

The given function f is

It is evident that f is defined at all points of the real line.

Let c be a real number.

Case I:

$$\text{If } c \neq 0, \text{ then } f(c) = c^2 \sin \frac{1}{c}$$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} \left(x^2 \sin \frac{1}{x} \right) = \left(\lim_{x \rightarrow c} x^2 \right) \left(\lim_{x \rightarrow c} \sin \frac{1}{x} \right) = c^2 \sin \frac{1}{c}$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points $x \neq 0$

Case II:

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \left(x^2 \sin \frac{1}{x} \right) = \lim_{x \rightarrow 0} \left(x^2 \sin \frac{1}{x} \right)$$

It is known that, $-1 \leq \sin \frac{1}{x} \leq 1$, $x \neq 0$

$$\Rightarrow -x^2 \leq \sin \frac{1}{x} \leq x^2$$

$$\Rightarrow \lim_{x \rightarrow 0} (-x^2) \leq \lim_{x \rightarrow 0} \left(x^2 \sin \frac{1}{x} \right) \leq \lim_{x \rightarrow 0} x^2$$

$$\Rightarrow 0 \leq \lim_{x \rightarrow 0} \left(x^2 \sin \frac{1}{x} \right) \leq 0$$

$$\Rightarrow \lim_{x \rightarrow 0} \left(x^2 \sin \frac{1}{x} \right) = 0$$

$$\therefore \lim_{x \rightarrow 0^-} f(x) = 0$$

$$\text{Similarly, } \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \left(x^2 \sin \frac{1}{x} \right) = \lim_{x \rightarrow 0} \left(x^2 \sin \frac{1}{x} \right) = 0$$

$$\therefore \lim_{x \rightarrow 0} f(x) = f(0) = \lim_{x \rightarrow 0^+} f(x)$$

Therefore, f is continuous at $x = 0$

From the above observations, it can be concluded that f is continuous at every point of the real line.

Thus, f is a continuous function.

Question 25:

Examine the continuity of f , where f is defined by

$$f(x) = \begin{cases} \sin x - \cos x, & \text{if } x \neq 0 \\ -1 & \text{if } x = 0 \end{cases}$$

Answer

$$f(x) = \begin{cases} \sin x - \cos x, & \text{if } x \neq 0 \\ -1 & \text{if } x = 0 \end{cases}$$

The given function f is

It is evident that f is defined at all points of the real line.

Let c be a real number.

Case I:

If $c \neq 0$, then $f(c) = \sin c - \cos c$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (\sin x - \cos x) = \sin c - \cos c$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x \neq 0$

Case II:

If $c = 0$, then $f(0) = -1$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0} (\sin x - \cos x) = \sin 0 - \cos 0 = 0 - 1 = -1$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0} (\sin x - \cos x) = \sin 0 - \cos 0 = 0 - 1 = -1$$

$$\therefore \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = f(0)$$

Therefore, f is continuous at $x = 0$

From the above observations, it can be concluded that f is continuous at every point of the real line.

Thus, f is a continuous function.

Question 26:

Find the values of k so that the function f is continuous at the indicated point.

$$f(x) = \begin{cases} \frac{k \cos x}{\pi - 2x}, & \text{if } x \neq \frac{\pi}{2} \\ 3, & \text{if } x = \frac{\pi}{2} \end{cases} \quad \text{at } x = \frac{\pi}{2}$$

Answer

$$f(x) = \begin{cases} \frac{k \cos x}{\pi - 2x}, & \text{if } x \neq \frac{\pi}{2} \\ 3, & \text{if } x = \frac{\pi}{2} \end{cases}$$

The given function f is

The given function f is continuous at

It is evident that f is defined at

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = f(2)$$

$$\Rightarrow \lim_{x \rightarrow 2^-} (kx^2) = \lim_{x \rightarrow 2^+} (3) = 4k$$

$$\Rightarrow k \times 2^2 = 3 = 4k$$

$$\Rightarrow 4k = 3 = 4k$$

$$\Rightarrow 4k = 3$$

$$\Rightarrow k = \frac{3}{4}$$

Therefore, the required value of

Question 29:

Find the values of k so that the function f is continuous at the indicated point.

$$f(x) = \begin{cases} kx+1, & \text{if } x \leq 5 \\ 3x-5, & \text{if } x > 5 \end{cases} \quad \text{at } x = 5$$

Answer

$$f(x) = \begin{cases} kx+1, & \text{if } x \leq 5 \\ 3x-5, & \text{if } x > 5 \end{cases}$$

The given function f is

The given function f is continuous at $x = 5$, if f is defined at $x = 5$ and if the value of f at $x = 5$ equals the limit of f at $x = 5$

It is evident that f is defined at $x = 5$ and

Answer

$$f(x) = \begin{cases} 5, & \text{if } x \leq 2 \\ ax+b, & \text{if } 2 < x < 10 \\ 21, & \text{if } x \geq 10 \end{cases}$$

The given function f is

It is evident that the given function f is defined at all points of the real line.

If f is a continuous function, then f is continuous at all real numbers.

In particular, f is continuous at $x = 2$ and $x = 10$

Since f is continuous at $x = 2$, we obtain

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = f(2)$$

$$\Rightarrow \lim_{x \rightarrow 2^-} (5) = \lim_{x \rightarrow 2^+} (ax+b) = 5$$

$$\Rightarrow 5 = 2a + b = 5$$

$$\Rightarrow 2a + b = 5$$

...(1)

Since f is continuous at $x = 10$, we obtain

$$\lim_{x \rightarrow 10^-} f(x) = \lim_{x \rightarrow 10^+} f(x) = f(10)$$

$$\Rightarrow \lim_{x \rightarrow 10^-} (ax+b) = \lim_{x \rightarrow 10^+} (21) = 21$$

$$\Rightarrow 10a + b = 21 = 21$$

$$\Rightarrow 10a + b = 21$$

...(2)

On subtracting equation (1) from equation (2), we obtain

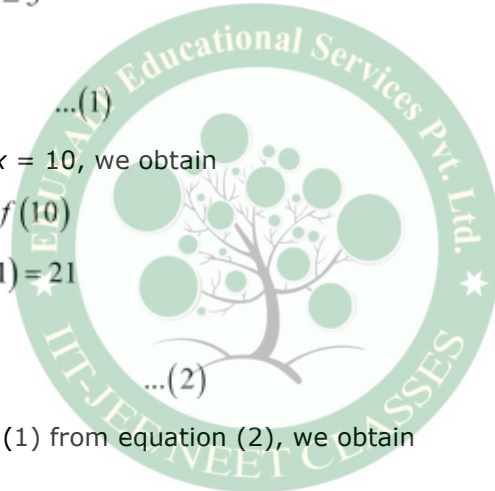
$$8a = 16$$

$$\Rightarrow a = 2$$

By putting $a = 2$ in equation (1), we obtain

$$2 \times 2 + b = 5$$

$$\Rightarrow 4 + b = 5$$



$$\Rightarrow b = 1$$

Therefore, the values of a and b for which f is a continuous function are 2 and 1 respectively.

Question 31:

Show that the function defined by $f(x) = \cos(x^2)$ is a continuous function.

Answer

The given function is $f(x) = \cos(x^2)$

This function f is defined for every real number and f can be written as the composition of two functions as,

$f = g \circ h$, where $g(x) = \cos x$ and $h(x) = x^2$

$$\left[\because (g \circ h)(x) = g(h(x)) = g(x^2) = \cos(x^2) = f(x) \right]$$

It has to be first proved that $g(x) = \cos x$ and $h(x) = x^2$ are continuous functions.

It is evident that g is defined for every real number.

Let c be a real number.

Then, $g(c) = \cos c$

Put $x = c + h$

If $x \rightarrow c$, then $h \rightarrow 0$

$$\begin{aligned} \lim_{x \rightarrow c} g(x) &= \lim_{x \rightarrow c} \cos x \\ &= \lim_{h \rightarrow 0} \cos(c + h) \\ &= \lim_{h \rightarrow 0} [\cos c \cos h - \sin c \sin h] \\ &= \lim_{h \rightarrow 0} \cos c \cos h - \lim_{h \rightarrow 0} \sin c \sin h \\ &= \cos c \cos 0 - \sin c \sin 0 \\ &= \cos c \times 1 - \sin c \times 0 \\ &= \cos c \end{aligned}$$

$$\therefore \lim_{x \rightarrow c} g(x) = g(c)$$

Therefore, $g(x) = \cos x$ is continuous function.

$$h(x) = x^2$$

Clearly, h is defined for every real number.

Let k be a real number, then $h(k) = k^2$

$$\lim_{x \rightarrow k} h(x) = \lim_{x \rightarrow k} x^2 = k^2$$

$$\therefore \lim_{x \rightarrow k} h(x) = h(k)$$

Therefore, h is a continuous function.

It is known that for real valued functions g and h , such that $(g \circ h)$ is defined at c , if g is continuous at c and if h is continuous at $g(c)$, then $(g \circ h)$ is continuous at c .

Therefore, $f(x) = (g \circ h)(x) = \cos(x^2)$ is a continuous function.

Question 32:

Show that the function defined by

Therefore, g is continuous at all points x , such that $x < 0$

Case II:

If $c > 0$, then $g(c) = c$ and $\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} x = c$

$$\therefore \lim_{x \rightarrow c} g(x) = g(c)$$

Therefore, g is continuous at all points x , such that $x > 0$

Case III:

If $c = 0$, then $g(c) = g(0) = 0$

$$\lim_{x \rightarrow 0^-} g(x) = \lim_{x \rightarrow 0^-} (-x) = 0$$

$$\lim_{x \rightarrow 0^+} g(x) = \lim_{x \rightarrow 0^+} (x) = 0$$

$$\therefore \lim_{x \rightarrow 0} g(x) = \lim_{x \rightarrow 0} (x) = g(0)$$

Therefore, g is continuous at $x = 0$

From the above three observations, it can be concluded that g is continuous at all points.

$$h(x) = \cos x$$

It is evident that $h(x) = \cos x$ is defined for every real number.

Let c be a real number. Put $x = c + h$

If $x \rightarrow c$, then $h \rightarrow 0$

$$h(c) = \cos c$$

$$\begin{aligned} \lim_{x \rightarrow c} h(x) &= \lim_{x \rightarrow c} \cos x \\ &= \lim_{h \rightarrow 0} \cos(c + h) \\ &= \lim_{h \rightarrow 0} [\cos c \cos h - \sin c \sin h] \\ &= \lim_{h \rightarrow 0} \cos c \cos h - \lim_{h \rightarrow 0} \sin c \sin h \\ &= \cos c \cos 0 - \sin c \sin 0 \\ &= \cos c \times 1 - \sin c \times 0 \\ &= \cos c \end{aligned}$$

$$\therefore \lim_{x \rightarrow c} h(x) = h(c)$$

Therefore, $h(x) = \cos x$ is a continuous function.

It is known that for real valued functions g and h , such that $(g \circ h)$ is defined at c , if g is continuous at c and if f is continuous at $g(c)$, then $(f \circ g)$ is continuous at c .

Therefore, $f(x) = (goh)(x) = g(h(x)) = g(\cos x) = |\cos x|$ is a continuous function.

Question 33:

Answer

$$\lim_{x \rightarrow 0^-} g(x) = \lim_{x \rightarrow 0^-} (-x) = 0$$

$$\lim_{x \rightarrow 0^+} g(x) = \lim_{x \rightarrow 0^+} (x) = 0$$

$$\therefore \lim_{x \rightarrow 0} g(x) = \lim_{x \rightarrow 0} (x) = g(0)$$

Therefore, g is continuous at $x = 0$

From the above three observations, it can be concluded that g is continuous at all points.

$$h(x) = \sin x$$

It is evident that $h(x) = \sin x$ is defined for every real number.

Let c be a real number. Put $x = c + k$

If $x \rightarrow c$, then $k \rightarrow 0$

$$h(c) = \sin c$$

$$h(c) = \sin c$$

$$\lim_{x \rightarrow c} h(x) = \lim_{x \rightarrow c} \sin x$$

$$= \lim_{k \rightarrow 0} \sin(c + k)$$

$$= \lim_{k \rightarrow 0} [\sin c \cos k + \cos c \sin k]$$

$$= \lim_{k \rightarrow 0} (\sin c \cos k) + \lim_{k \rightarrow 0} (\cos c \sin k)$$

$$= \sin c \cos 0 + \cos c \sin 0$$

$$= \sin c + 0$$

$$= \sin c$$

$$\therefore \lim_{x \rightarrow c} h(x) = g(c)$$

Therefore, h is a continuous function.

It is known that for real valued functions g and h , such that $(g \circ h)$ is defined at c , if g is continuous at c and if h is continuous at $g(c)$, then $(g \circ h)$ is continuous at c .

Therefore, $f(x) = (g \circ h)(x) = g(h(x)) = g(\sin x) = |\sin x|$ is a continuous function.

Question 34:

Answer

The given function is

The two functions, g and h , are defined as

$$g(x) = |x| \text{ and } h(x) = |x+1|$$

Then, $f = g - h$

The continuity of g and h is examined first.

$g(x) = |x|$ can be written as

$$g(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

Clearly, g is defined for all real numbers.

Let c be a real number.

Case I:

If $c < 0$, then $g(c) = -c$ and $\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} (-x) = -c$

$$\therefore \lim_{x \rightarrow c} g(x) = g(c)$$

Therefore, g is continuous at all points x , such that $x < 0$

Case II:

If $c > 0$, then $g(c) = c$ and $\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} x = c$

$$\therefore \lim_{x \rightarrow c} g(x) = g(c)$$

Therefore, g is continuous at all points x , such that $x > 0$

Case III:

If $c = 0$, then $g(c) = g(0) = 0$

$$\lim_{x \rightarrow 0^-} g(x) = \lim_{x \rightarrow 0^-} (-x) = 0$$

$$\lim_{x \rightarrow 0^+} g(x) = \lim_{x \rightarrow 0^+} (x) = 0$$

$$\therefore \lim_{x \rightarrow 0} g(x) = \lim_{x \rightarrow 0} (x) = g(0)$$

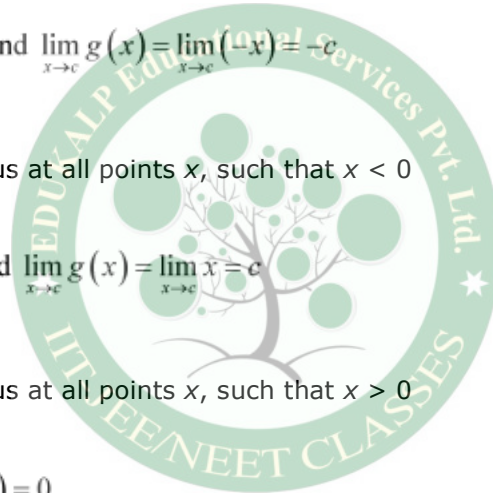
Therefore, g is continuous at $x = 0$

From the above three observations, it can be concluded that g is continuous at all points.

$h(x) = |x+1|$ can be written as

$$h(x) = \begin{cases} -(x+1), & \text{if } x < -1 \\ x+1, & \text{if } x \geq -1 \end{cases}$$

Clearly, h is defined for every real number.



Let c be a real number.

Case I:

If $c < -1$, then $h(c) = -(c+1)$ and $\lim_{x \rightarrow c} h(x) = \lim_{x \rightarrow c} [-(x+1)] = -(c+1)$

$$\therefore \lim_{x \rightarrow c} h(x) = h(c)$$

Therefore, h is continuous at all points x , such that $x < -1$

Case II:

If $c > -1$, then $h(c) = c+1$ and $\lim_{x \rightarrow c} h(x) = \lim_{x \rightarrow c} (x+1) = c+1$

$$\therefore \lim_{x \rightarrow c} h(x) = h(c)$$

Therefore, h is continuous at all points x , such that $x > -1$

Case III:

If $c = -1$, then $h(c) = h(-1) = -1+1 = 0$

$$\lim_{x \rightarrow -1^-} h(x) = \lim_{x \rightarrow -1^-} [-(x+1)] = -(-1+1) = 0$$

$$\lim_{x \rightarrow -1^+} h(x) = \lim_{x \rightarrow -1^+} (x+1) = (-1+1) = 0$$

$$\therefore \lim_{x \rightarrow -1^-} h(x) = \lim_{h \rightarrow -1^+} h(x) = h(-1)$$

Therefore, h is continuous at $x = -1$

From the above three observations, it can be concluded that h is continuous at all points of the real line.

g and h are continuous functions. Therefore, $f = g - h$ is also a continuous function.

Therefore, f has no point of discontinuity.

