Exercise 5.2

Question 1: $\sin(x^2 + 5)$

Differentiate the functions with respect to *x*.

Answer

Let $f(x) = \sin(x^2 + 5)$, $u(x) = x^2 + 5$, and $v(t) = \sin t$

Then, $(vou)(x) = v(u(x)) = v(x^2 + 5) = \tan(x^2 + 5) = f(x)$

Thus, *f* is a composite of two functions.

Put $t = u(x) = x^2 + 5$

Then, we obtain

$$\frac{dv}{dt} = \frac{d}{dt}(\sin t) = \cos t = \cos(x^2 + 5)$$

$$\frac{dt}{dx} = \frac{d}{dx}(x^2 + 5) = \frac{d}{dx}(x^2) + \frac{d}{dx}(5) = 2x + 0 = 2x$$

Therefore, by chain rule, $\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = \cos(x^2 + 5) \times 2x = 2x\cos(x^2 + 5)$

Alternate method

$$\frac{d}{dx}\left[\sin(x^2 + 5)\right] = \cos(x^2 + 5) \cdot \frac{d}{dx}(x^2 + 5)$$

$$= \cos(x^2 + 5) \cdot \left[\frac{d}{dx}(x^2) + \frac{d}{dx}(5)\right]$$

$$= \cos(x^2 + 5) \cdot [2x + 0]$$

$$= 2x\cos(x^2 + 5)$$

Question 2: $\cos(\sin x)$

Differentiate the functions with respect to *x*.

Answer

Let $f(x)=\cos(\sin x), u(x)=\sin x,$ and $v(t)=\cos t$

Then, $(vou)(x)=v(u(x))=v(\sin x)=\cos(\sin x)=f(x)$

Thus, *f* is a composite function of two functions.

Put *t* = *u* (*x*) = sin *x*

$\therefore \frac{dv}{dt}=\frac{d}{dt}[\cos t]=-\sin t=-\sin(\sin x)$

$\frac{dt}{dx}=\frac{d}{dx}(\sin x)=\cos x$

By chain rule, $\frac{df}{dx}=\frac{dv}{dt}\cdot\frac{dt}{dx}=-\sin(\sin x)\cdot\cos x=-\cos x\sin(\sin x)$

Alternate method

$\frac{d}{dx}\left[\cos(\sin x)\right]=-\sin(\sin x)\cdot\frac{d}{dx}(\sin x)=-\sin(\sin x)\cdot\cos x=-\cos x\sin(\sin x)$

Question 3: $\sin(ax+b)$

Differentiate the functions with respect to *x*.

Answer

Let $f(x)=\sin(ax+b),\ u(x)=ax+b,$ and $v(t)=\sin t$

Then, $(vou)(x)=v(u(x))=v(ax+b)=\sin(ax+b)=f(x)$

Thus, *f* is a composite function of two functions, *u* and *v*.

Put *t* = *u* (*x*) = *ax* + *b*

Therefore,

$\frac{dv}{dt}=\frac{d}{dt}(\sin t)=\cos t=\cos(ax+b)$

$\frac{dt}{dx}=\frac{d}{dx}(ax+b)=\frac{d}{dx}(ax)+\frac{d}{dx}(b)=a+0=a$

Hence, by chain rule, we obtain

$\frac{df}{dx}=\frac{dv}{dt}\cdot\frac{dt}{dx}=\cos(ax+b)\cdot a=a\cos(ax+b)$

Alternate method

$$\frac{d}{dx}\left[\sin(ax+b)\right]=\cos(ax+b).\frac{d}{dx}(ax+b)$$
$$=\cos(ax+b)\cdot\left[\frac{d}{dx}(ax)+\frac{d}{dx}(b)\right]$$
$$=\cos(ax+b).(a+0)$$
$$=a\cos(ax+b)$$

Question 4: $\sec(\tan(\sqrt{x}))$

Differentiate the functions with respect to *x*.

Answer

$$\text{Let } f(x)=\sec\left(\tan\sqrt{x}\right), u(x)=\sqrt{x}, v(t)=\tan t, \text{ and } w(s)=\sec s$$

$$\text{Then, } (wovou)(x)=w\left[v\left(u(x)\right)\right]=w\left[v\left(\sqrt{x}\right)\right]=w\left(\tan\sqrt{x}\right)=\sec\left(\tan\sqrt{x}\right)=f(x)$$

Thus, *f* is a composite function of three functions, *u*, *v*, and *w*.

$$\text{Put } s=v(t)=\tan t \text{ and } t=u(x)=\sqrt{x}$$

$$\text{Then, } \frac{dw}{ds}=\frac{d}{ds}(\sec s)=\sec s\tan s=\sec(\tan t).\tan(\tan t) \quad [s=\tan t]$$
$$=\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right) \quad \left[t=\sqrt{x}\right]$$

$$\frac{ds}{dt}=\frac{d}{dt}(\tan t)=\sec^2 t=\sec^2\sqrt{x}$$

$$\frac{dt}{dx}=\frac{d}{dx}\left(\sqrt{x}\right)=\frac{d}{dx}\left(x^{\frac{1}{2}}\right)=\frac{1}{2}\cdot x^{\frac{1}{2}-1}=\frac{1}{2\sqrt{x}}$$

Hence, by chain rule, we obtain

$$\frac{dt}{dx}=\frac{dw}{ds}\cdot\frac{ds}{dt}\cdot\frac{dt}{dx}$$
$$=\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right)\times\sec^2\sqrt{x}\times\frac{1}{2\sqrt{x}}$$
$$=\frac{1}{2\sqrt{x}}\sec^2\sqrt{x}\sec\left(\tan\sqrt{x}\right)\tan\left(\tan\sqrt{x}\right)$$
$$=\frac{\sec^2\sqrt{x}\sec\left(\tan\sqrt{x}\right)\tan\left(\tan\sqrt{x}\right)}{2\sqrt{x}}$$

Alternate method

$$\frac{d}{dx}\left[\sec\left(\tan\sqrt{x}\right)\right]=\sec\left(\tan\sqrt{x}\right).\tan\left(\tan\sqrt{x}\right)\cdot\frac{d}{dx}\left(\tan\sqrt{x}\right)$$
$$=\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right)\cdot\sec^2\left(\sqrt{x}\right)\cdot\frac{d}{dx}\left(\sqrt{x}\right)$$
$$=\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right)\cdot\sec^2\left(\sqrt{x}\right)\cdot\frac{1}{2\sqrt{x}}$$
$$=\frac{\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right)\sec^2\left(\sqrt{x}\right)}{2\sqrt{x}}$$

Question 5: $\frac{\sin(ax+b)}{\cos(cx+d)}$

Differentiate the functions with respect to *x*.

Answer

The given function is $f(x)=\frac{\sin(ax+b)}{\cos(cx+d)}=\frac{g(x)}{h(x)}$, where $g(x) = \sin(ax + b)$ and $h(x) = \cos(cx + d)$

$$\therefore f'=\frac{g'h-gh'}{h^2}$$

Consider $g(x)=\sin(ax+b)$

Let $u(x)=ax+b, v(t)=\sin t$

Then, $(vou)(x)=v(u(x))=v(ax+b)=\sin(ax+b)=g(x)$

$\therefore$ g is a composite function of two functions, u and v.

Put $t = u(x) = ax + b$

$$\frac{dv}{dt} = \frac{d}{dt}(\sin t) = \cos t = \cos(ax+b)$$

$$\frac{dt}{dx} = \frac{d}{dx}(ax+b) = \frac{d}{dx}(ax) + \frac{d}{dx}(b) = a + 0 = a$$

Therefore, by chain rule, we obtain

$$g' = \frac{dg}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = \cos(ax+b) \cdot a = a\cos(ax+b)$$

Consider $h(x) = \cos(cx+d)$

Let $p(x) = cx + d,\ q(y) = \cos y$

Then, $(qop)(x) = q(p(x)) = q(cx+d) = \cos(cx+d) = h(x)$

$\therefore h$ is a composite function of two functions, p and q.

Put $y = p(x) = cx + d$

$$\frac{dq}{dy} = \frac{d}{dy}(\cos y) = -\sin y = -\sin(cx+d)$$

$$\frac{dy}{dx} = \frac{d}{dx}(cx+d) = \frac{d}{dx}(cx) + \frac{d}{dx}(d) = c$$

Therefore, by chain rule, we obtain

$$h' = \frac{dh}{dx} = \frac{dq}{dy} \cdot \frac{dy}{dx} = -\sin(cx+d) \times c = -c\sin(cx+d)$$

$$\therefore f' = \frac{a\cos(ax+b)\cdot\cos(cx+d) - \sin(ax+b)\{-c\sin(cx+d)\}}{[\cos(cx+d)]^2}$$
$$= \frac{a\cos(ax+b)}{\cos(cx+d)} + c\sin(ax+b)\cdot\frac{\sin(cx+d)}{\cos(cx+d)}\times\frac{1}{\cos(cx+d)}$$
$$= a\cos(ax+b)\sec(cx+d) + c\sin(ax+b)\tan(cx+d)\sec(cx+d)$$

Question 6: $\cos x^3 \cdot \sin^2(x^5)$

Differentiate the functions with respect to *x*.

Answer

The given function is .

$$\frac{d}{dx}\left[\cos x^3\cdot\sin^2(x^5)\right] = \sin^2(x^5)\times\frac{d}{dx}(\cos x^3) + \cos x^3\times\frac{d}{dx}\left[\sin^2(x^5)\right]$$
$$= \sin^2(x^5)\times(-\sin x^3)\times\frac{d}{dx}(x^3) + \cos x^3\times 2\sin(x^5)\cdot\frac{d}{dx}\left[\sin x^5\right]$$
$$= -\sin x^3\sin^2(x^5)\times 3x^2 + 2\sin x^5\cos x^3\cdot\cos x^5\times\frac{d}{dx}(x^5)$$
$$= -3x^2\sin x^3\cdot\sin^2(x^5) + 2\sin x^5\cos x^5\cos x^3\cdot\times 5x^4$$
$$= 10x^4\sin x^5\cos x^5\cos x^3 - 3x^2\sin x^3\sin^2(x^5)$$

Question 7: $2\sqrt{\cot(x^2)}$

Differentiate the functions with respect to *x*.

Answer

$$\frac{d}{dx}\left[2\sqrt{\cot(x^2)}\right]$$
$$= 2\cdot\frac{1}{2\sqrt{\cot(x^2)}}\times\frac{d}{dx}\left[\cot(x^2)\right]$$
$$= \sqrt{\frac{\sin(x^2)}{\cos(x^2)}}\times -\operatorname{cosec}^2(x^2)\times\frac{d}{dx}(x^2)$$
$$= -\sqrt{\frac{\sin(x^2)}{\cos(x^2)}}\times\frac{1}{\sin^2(x^2)}\times(2x)$$
$$= \frac{-2x}{\sqrt{\cos x^2}\sqrt{\sin x^2}\sin x^2}$$
$$= \frac{-2\sqrt{2}\,x}{\sqrt{2\sin x^2\cos x^2}\sin x^2}$$
$$= \frac{-2\sqrt{2}\,x}{\sin x^2\sqrt{\sin 2x^2}}$$

Question 8: $\cos(\sqrt{x})$

Differentiate the functions with respect to *x*.

Answer

Let $f(x) = \cos(\sqrt{x})$

Also, let $u(x) = \sqrt{x}$

And, $v(t) = \cos t$

Then, $(vou)(x) = v(u(x))$

$$= v(\sqrt{x})$$
$$= \cos\sqrt{x}$$
$$= f(x)$$

Clearly, *f* is a composite function of two functions, *u* and *v*, such that

Then, $\frac{dt}{dx}=\frac{d}{dx}\left(\sqrt{x}\right)=\frac{d}{dx}\left(x^{\frac{1}{2}}\right)=\frac{1}{2}x^{-\frac{1}{2}}$

$=\frac{1}{2\sqrt{x}}$

And, $\frac{dv}{dt}=\frac{d}{dt}(\cos t)=-\sin t$

$=-\sin\left(\sqrt{x}\right)$

By using chain rule, we obtain

$$\frac{dt}{dx}=\frac{dv}{dt}\cdot\frac{dt}{dx}$$

$$=-\sin\left(\sqrt{x}\right)\cdot\frac{1}{2\sqrt{x}}$$

$$=-\frac{1}{2\sqrt{x}}\sin\left(\sqrt{x}\right)$$

$$=-\frac{\sin\left(\sqrt{x}\right)}{2\sqrt{x}}$$

Alternate method

$$\frac{d}{dx}\left[\cos\left(\sqrt{x}\right)\right]=-\sin\left(\sqrt{x}\right)\cdot\frac{d}{dx}\left(\sqrt{x}\right)$$

$$=-\sin\left(\sqrt{x}\right)\times\frac{d}{dx}\left(x^{\frac{1}{2}}\right)$$

$$=-\sin\sqrt{x}\times\frac{1}{2}x^{-\frac{1}{2}}$$

$$=\frac{-\sin\sqrt{x}}{2\sqrt{x}}$$

Question 9: Prove that the function f given by

$$f(x) = |x-1|,\ x\in \mathbf{R}$$

is not differentiable at $x = 1$.

Answer

The given function is

It is known that a function *f* is differentiable at a point *x* = *c* in its domain if both

$\lim_{h\to 0^-}\frac{f(c+h)-f(c)}{h}$ and $\lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h}$ are finite and equal.

To check the differentiability of the given function at *x* = 1,

consider the left hand limit of *f* at *x* = 1

$$\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^-}\frac{|1+h-1|-|1-1|}{h}$$

$$=\lim_{h\to 0^-}\frac{|h|-0}{h}=\lim_{h\to 0^-}\frac{-h}{h} \qquad (h<0\Rightarrow |h|=-h)$$

$$=-1$$

Consider the right hand limit of f at $x=1$

$$\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^+}\frac{|1+h-1|-|1-1|}{h}$$

$$=\lim_{h\to 0^+}\frac{|h|-0}{h}=\lim_{h\to 0^+}\frac{h}{h} \qquad (h>0\Rightarrow |h|=h)$$

$$=1$$

Since the left and right hand limits of *f* at *x* = 1 are not equal, *f* is not differentiable at *x* = 1

Question 10:

Prove that the greatest integer function defined by

$$f(x) = [x],\ 0 < x < 3$$

is not differentiable at $x = 1$ and $x = 2$.

The given function *f* is

It is known that a function *f* is differentiable at a point *x* = *c* in its domain if both

$\lim_{h\to 0^-}\frac{f(c+h)-f(c)}{h}$ and $\lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h}$ are finite and equal.

To check the differentiability of the given function at *x* = 1, consider the left hand limit of *f* at *x* = 1

$$\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^-}\frac{[1+h]-[1]}{h}$$

$$=\lim_{h\to 0^-}\frac{0-1}{h}=\lim_{h\to 0^-}\frac{-1}{h}=\infty$$

Consider the right hand limit of f at $x=1$

$$\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^+}\frac{[1+h]-[1]}{h}$$

$$=\lim_{h\to 0^+}\frac{1-1}{h}=\lim_{h\to 0^+}0=0$$

Since the left and right hand limits of f at x = 1 are not equal, f is not differentiable at x = 1

To check the differentiability of the given function at x = 2, consider the left hand limit of f at x = 2

$$\lim_{h\to 0^-}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0^-}\frac{[2+h]-[2]}{h}$$

$$=\lim_{h\to 0^-}\frac{1-2}{h}=\lim_{h\to 0^-}\frac{-1}{h}=\infty$$

Consider the right hand limit of f at $x=1$

$$\lim_{h\to 0^+}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0^+}\frac{[2+h]-[2]}{h}$$

$$=\lim_{h\to 0^+}\frac{2-2}{h}=\lim_{h\to 0^+}0=0$$

Since the left and right hand limits of f at x = 2 are not equal, f is not differentiable at x = 2