

Exercise 5.5

Question 1:

Differentiate the function with respect to x .

$$\cos x \cdot \cos 2x \cdot \cos 3x$$

Answer

$$\text{Let } y = \cos x \cdot \cos 2x \cdot \cos 3x$$

Taking logarithm on both the sides, we obtain

$$\log y = \log (\cos x \cdot \cos 2x \cdot \cos 3x)$$

$$\Rightarrow \log y = \log (\cos x) + \log (\cos 2x) + \log (\cos 3x)$$

Differentiating both sides with respect to x , we obtain

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{\cos x} \cdot \frac{d}{dx} (\cos x) + \frac{1}{\cos 2x} \cdot \frac{d}{dx} (\cos 2x) + \frac{1}{\cos 3x} \cdot \frac{d}{dx} (\cos 3x)$$

$$\Rightarrow \frac{dy}{dx} = y \left[-\frac{\sin x}{\cos x} - \frac{\sin 2x}{\cos 2x} \cdot \frac{d}{dx} (2x) - \frac{\sin 3x}{\cos 3x} \cdot \frac{d}{dx} (3x) \right]$$

$$\therefore \frac{dy}{dx} = -\cos x \cdot \cos 2x \cdot \cos 3x [\tan x + 2 \tan 2x + 3 \tan 3x]$$

Question 2:

Differentiate the function with respect to x .

$$\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$$

Answer

$$\text{Let } y = \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$$

Taking logarithm on both the sides, we obtain

$$\begin{aligned}\log y &= \log \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}} \\ \Rightarrow \log y &= \frac{1}{2} \log \left[\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)} \right] \\ \Rightarrow \log y &= \frac{1}{2} \left[\log \{(x-1)(x-2)\} - \log \{(x-3)(x-4)(x-5)\} \right] \\ \Rightarrow \log y &= \frac{1}{2} \left[\log(x-1) + \log(x-2) - \log(x-3) - \log(x-4) - \log(x-5) \right]\end{aligned}$$

Differentiating both sides with respect to x , we obtain

$$\begin{aligned}\frac{1}{y} \frac{dy}{dx} &= \frac{1}{2} \left[\frac{1}{x-1} \cdot \frac{d}{dx}(x-1) + \frac{1}{x-2} \cdot \frac{d}{dx}(x-2) - \frac{1}{x-3} \cdot \frac{d}{dx}(x-3) \right. \\ &\quad \left. - \frac{1}{x-4} \cdot \frac{d}{dx}(x-4) - \frac{1}{x-5} \cdot \frac{d}{dx}(x-5) \right] \\ \Rightarrow \frac{dy}{dx} &= \frac{y}{2} \left(\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5} \right) \\ \therefore \frac{dy}{dx} &= \frac{1}{2} \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}} \left[\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5} \right]\end{aligned}$$

Question 3:

Differentiate the function with respect to x .

$$(\log x)^{\cos x}$$

Answer

$$\text{Let } y = (\log x)^{\cos x}$$

Taking logarithm on both the sides, we obtain

$$\log y = \cos x \cdot \log(\log x)$$

Differentiating both sides with respect to x , we obtain

$$\begin{aligned}
 \frac{1}{y} \cdot \frac{dy}{dx} &= \frac{d}{dx} (\cos x) \times \log(\log x) + \cos x \times \frac{d}{dx} [\log(\log x)] \\
 \Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} &= -\sin x \log(\log x) + \cos x \times \frac{1}{\log x} \cdot \frac{d}{dx} (\log x) \\
 \Rightarrow \frac{dy}{dx} &= y \left[-\sin x \log(\log x) + \frac{\cos x}{\log x} \times \frac{1}{x} \right] \\
 \therefore \frac{dy}{dx} &= (\log x)^{\cos x} \left[\frac{\cos x}{x \log x} - \sin x \log(\log x) \right]
 \end{aligned}$$

Question 4:

Differentiate the function with respect to x .

$$x^x - 2^{\sin x}$$

Answer

$$\text{Let } y = x^x - 2^{\sin x}$$

$$\text{Also, let } x^x = u \text{ and } 2^{\sin x} = v$$

$$\therefore y = u - v$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}$$

$$u = x^x$$

Taking logarithm on both the sides, we obtain

$$\log u = x \log x$$

Differentiating both sides with respect to x , we obtain

$$\frac{1}{u} \frac{du}{dx} = \left[\frac{d}{dx} (x) \times \log x + x \times \frac{d}{dx} (\log x) \right]$$

$$\Rightarrow \frac{du}{dx} = u \left[1 \times \log x + x \times \frac{1}{x} \right]$$

$$\Rightarrow \frac{du}{dx} = x^x (\log x + 1)$$

$$\Rightarrow \frac{du}{dx} = x^x (1 + \log x)$$

$$v = 2^{\sin x}$$

Taking logarithm on both the sides with respect to x , we obtain



$$\log v = \sin x \cdot \log 2$$

Differentiating both sides with respect to x , we obtain

$$\frac{1}{v} \cdot \frac{dv}{dx} = \log 2 \cdot \frac{d}{dx}(\sin x)$$

$$\Rightarrow \frac{dv}{dx} = v \log 2 \cos x$$

$$\Rightarrow \frac{dv}{dx} = 2^{\sin x} \cos x \log 2$$

$$\therefore \frac{dy}{dx} = x^x (1 + \log x) - 2^{\sin x} \cos x \log 2$$

Question 5:

Differentiate the function with respect to x .

$$(x+3)^2 \cdot (x+4)^3 \cdot (x+5)^4$$

Answer

$$\text{Let } y = (x+3)^2 \cdot (x+4)^3 \cdot (x+5)^4$$

Taking logarithm on both the sides, we obtain

$$\log y = \log (x+3)^2 + \log (x+4)^3 + \log (x+5)^4$$

$$\Rightarrow \log y = 2 \log (x+3) + 3 \log (x+4) + 4 \log (x+5)$$

Differentiating both sides with respect to x , we obtain

$$\frac{1}{y} \cdot \frac{dy}{dx} = 2 \cdot \frac{1}{x+3} \cdot \frac{d}{dx}(x+3) + 3 \cdot \frac{1}{x+4} \cdot \frac{d}{dx}(x+4) + 4 \cdot \frac{1}{x+5} \cdot \frac{d}{dx}(x+5)$$

$$\Rightarrow \frac{dy}{dx} = y \left[\frac{2}{x+3} + \frac{3}{x+4} + \frac{4}{x+5} \right]$$

$$\Rightarrow \frac{dy}{dx} = (x+3)^2 (x+4)^3 (x+5)^4 \cdot \left[\frac{2}{x+3} + \frac{3}{x+4} + \frac{4}{x+5} \right]$$

$$\Rightarrow \frac{dy}{dx} = (x+3)^2 (x+4)^3 (x+5)^4 \cdot \left[\frac{2(x+4)(x+5) + 3(x+3)(x+5) + 4(x+3)(x+4)}{(x+3)(x+4)(x+5)} \right]$$

$$\Rightarrow \frac{dy}{dx} = (x+3)(x+4)^2 (x+5)^3 \cdot [2(x^2 + 9x + 20) + 3(x^2 + 8x + 15) + 4(x^2 + 7x + 12)]$$

$$\therefore \frac{dy}{dx} = (x+3)(x+4)^2 (x+5)^3 (9x^2 + 70x + 133)$$

Question 6:

Differentiate the function with respect to x .

$$\left(x + \frac{1}{x}\right)^x + x^{\left(1 + \frac{1}{x}\right)}$$

Answer

$$\text{Let } y = \left(x + \frac{1}{x}\right)^x + x^{\left(1 + \frac{1}{x}\right)}$$

$$\text{Also, let } u = \left(x + \frac{1}{x}\right)^x \text{ and } v = x^{\left(1 + \frac{1}{x}\right)}$$

$$\therefore y = u + v$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$\text{Then, } u = \left(x + \frac{1}{x}\right)^x$$

$$\Rightarrow \log u = \log \left(x + \frac{1}{x}\right)^x$$

$$\Rightarrow \log u = x \log \left(x + \frac{1}{x}\right)$$

Differentiating both sides with respect to x , we obtain



$$\begin{aligned}
 \frac{1}{u} \cdot \frac{du}{dx} &= \frac{d}{dx} \left(x \right) \times \log \left(x + \frac{1}{x} \right) + x \times \frac{d}{dx} \left[\log \left(x + \frac{1}{x} \right) \right] \\
 \Rightarrow \frac{1}{u} \frac{du}{dx} &= 1 \times \log \left(x + \frac{1}{x} \right) + x \times \frac{1}{\left(x + \frac{1}{x} \right)} \cdot \frac{d}{dx} \left(x + \frac{1}{x} \right) \\
 \Rightarrow \frac{du}{dx} &= u \left[\log \left(x + \frac{1}{x} \right) + \frac{x}{\left(x + \frac{1}{x} \right)} \times \left(1 - \frac{1}{x^2} \right) \right] \\
 \Rightarrow \frac{du}{dx} &= \left(x + \frac{1}{x} \right)^x \left[\log \left(x + \frac{1}{x} \right) + \frac{\left(x - \frac{1}{x} \right)}{\left(x + \frac{1}{x} \right)} \right] \\
 \Rightarrow \frac{du}{dx} &= \left(x + \frac{1}{x} \right)^x \left[\log \left(x + \frac{1}{x} \right) + \frac{x^2 - 1}{x^2 + 1} \right] \\
 \Rightarrow \frac{du}{dx} &= \left(x + \frac{1}{x} \right)^x \left[\frac{x^2 - 1}{x^2 + 1} + \log \left(x + \frac{1}{x} \right) \right] \dots (2) \\
 v &= x^{\left(1 + \frac{1}{x} \right)} \\
 \Rightarrow \log v &= \log \left[x^{\left(1 + \frac{1}{x} \right)} \right] \\
 \Rightarrow \log v &= \left(1 + \frac{1}{x} \right) \log x
 \end{aligned}$$

Differentiating both sides with respect to x , we obtain

$$\begin{aligned}
 \frac{1}{v} \cdot \frac{dv}{dx} &= \left[\frac{d}{dx} \left(1 + \frac{1}{x} \right) \right] \times \log x + \left(1 + \frac{1}{x} \right) \cdot \frac{d}{dx} \log x \\
 \Rightarrow \frac{1}{v} \frac{dv}{dx} &= \left(-\frac{1}{x^2} \right) \log x + \left(1 + \frac{1}{x} \right) \cdot \frac{1}{x} \\
 \Rightarrow \frac{1}{v} \frac{dv}{dx} &= -\frac{\log x}{x^2} + \frac{1}{x} + \frac{1}{x^2} \\
 \Rightarrow \frac{dv}{dx} &= v \left[\frac{-\log x + x + 1}{x^2} \right] \\
 \Rightarrow \frac{dv}{dx} &= x^{\left(1 + \frac{1}{x}\right)} \left(\frac{x + 1 - \log x}{x^2} \right) \quad \dots(3)
 \end{aligned}$$

Therefore, from (1), (2), and (3), we obtain

$$\frac{dy}{dx} = \left(x + \frac{1}{x} \right)^x \left[\frac{x^2 - 1}{x^2 + 1} + \log \left(x + \frac{1}{x} \right) \right] + x^{\left(1 + \frac{1}{x}\right)} \left(\frac{x + 1 - \log x}{x^2} \right)$$

Question 7:

Differentiate the function with respect to x .

$$(\log x)^x + x^{\log x}$$

Answer

$$\text{Let } y = (\log x)^x + x^{\log x}$$

$$\text{Also, let } u = (\log x)^x \text{ and } v = x^{\log x}$$

$$\therefore y = u + v$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \dots(1)$$

$$u = (\log x)^x$$

$$\Rightarrow \log u = \log \left[(\log x)^x \right]$$

$$\Rightarrow \log u = x \log (\log x)$$

Differentiating both sides with respect to x , we obtain

$$\begin{aligned}
 \frac{1}{u} \frac{du}{dx} &= \frac{d}{dx} (x) \times \log(\log x) + x \cdot \frac{d}{dx} [\log(\log x)] \\
 \Rightarrow \frac{du}{dx} &= u \left[1 \times \log(\log x) + x \cdot \frac{1}{\log x} \cdot \frac{d}{dx} (\log x) \right] \\
 \Rightarrow \frac{du}{dx} &= (\log x)^x \left[\log(\log x) + \frac{x}{\log x} \cdot \frac{1}{x} \right] \\
 \Rightarrow \frac{du}{dx} &= (\log x)^x \left[\log(\log x) + \frac{1}{\log x} \right] \\
 \Rightarrow \frac{du}{dx} &= (\log x)^x \left[\frac{\log(\log x) \cdot \log x + 1}{\log x} \right] \\
 \Rightarrow \frac{du}{dx} &= (\log x)^{x-1} [1 + \log x \cdot \log(\log x)] \quad \dots(2)
 \end{aligned}$$

$$v = x^{\log x}$$

$$\Rightarrow \log v = \log(x^{\log x})$$

$$\Rightarrow \log v = \log x \log x = (\log x)^2$$

Differentiating both sides with respect to x , we obtain

$$\begin{aligned}
 \frac{1}{v} \cdot \frac{dv}{dx} &= \frac{d}{dx} [(\log x)^2] \\
 \Rightarrow \frac{1}{v} \cdot \frac{dv}{dx} &= 2(\log x) \cdot \frac{d}{dx} (\log x) \\
 \Rightarrow \frac{dv}{dx} &= 2v(\log x) \cdot \frac{1}{x} \\
 \Rightarrow \frac{dv}{dx} &= 2x^{\log x} \frac{\log x}{x} \\
 \Rightarrow \frac{dv}{dx} &= 2x^{\log x - 1} \cdot \log x \quad \dots(3)
 \end{aligned}$$

Therefore, from (1), (2), and (3), we obtain

$$\frac{dy}{dx} = (\log x)^{x-1} [1 + \log x \cdot \log(\log x)] + 2x^{\log x - 1} \cdot \log x$$

Question 8:

Differentiate the function with respect to x .

$$(\sin x)^x + \sin^{-1} \sqrt{x}$$

Answer

$$\text{Let } y = (\sin x)^x + \sin^{-1} \sqrt{x}$$

$$\text{Also, let } u = (\sin x)^x \text{ and } v = \sin^{-1} \sqrt{x}$$

$$\therefore y = u + v$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \dots(1)$$

$$u = (\sin x)^x$$

$$\Rightarrow \log u = \log (\sin x)^x$$

$$\Rightarrow \log u = x \log (\sin x)$$

Differentiating both sides with respect to x , we obtain

$$\Rightarrow \frac{1}{u} \frac{du}{dx} = \frac{d}{dx} (x) \times \log (\sin x) + x \times \frac{d}{dx} [\log (\sin x)]$$

$$\Rightarrow \frac{du}{dx} = u \left[1 \cdot \log (\sin x) + x \cdot \frac{1}{\sin x} \cdot \frac{d}{dx} (\sin x) \right]$$

$$\Rightarrow \frac{du}{dx} = (\sin x)^x \left[\log (\sin x) + \frac{x}{\sin x} \cdot \cos x \right]$$

$$\Rightarrow \frac{du}{dx} = (\sin x)^x (x \cot x + \log \sin x) \quad \dots(2)$$

$$v = \sin^{-1} \sqrt{x}$$

Differentiating both sides with respect to x , we obtain

$$\frac{dv}{dx} = \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{d}{dx} (\sqrt{x})$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}}$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{x-x^2}} \quad \dots(3)$$

Therefore, from (1), (2), and (3), we obtain

$$\frac{dy}{dx} = (\sin x)^x (x \cot x + \log \sin x) + \frac{1}{2\sqrt{x-x^2}}$$

Question 9:

Differentiate the function with respect to x .

$$x^{\sin x} + (\sin x)^{\cos x}$$

Answer

$$\text{Let } y = x^{\sin x} + (\sin x)^{\cos x}$$

$$\text{Also, let } u = x^{\sin x} \text{ and } v = (\sin x)^{\cos x}$$

$$\therefore y = u + v$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \dots(1)$$

$$u = x^{\sin x}$$

$$\Rightarrow \log u = \log(x^{\sin x})$$

$$\Rightarrow \log u = \sin x \log x$$

Differentiating both sides with respect to x , we obtain

$$\frac{1}{u} \frac{du}{dx} = \frac{d}{dx}(\sin x) \cdot \log x + \sin x \cdot \frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx} = u \left[\cos x \log x + \sin x \cdot \frac{1}{x} \right]$$

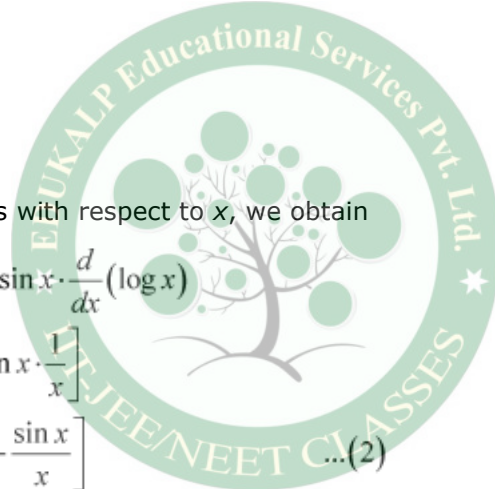
$$\Rightarrow \frac{du}{dx} = x^{\sin x} \left[\cos x \log x + \frac{\sin x}{x} \right] \quad \dots(2)$$

$$v = (\sin x)^{\cos x}$$

$$\Rightarrow \log v = \log(\sin x)^{\cos x}$$

$$\Rightarrow \log v = \cos x \log(\sin x)$$

Differentiating both sides with respect to x , we obtain



$$\begin{aligned}
 \frac{1}{v} \frac{dv}{dx} &= \frac{d}{dx} (\cos x) \times \log(\sin x) + \cos x \times \frac{d}{dx} [\log(\sin x)] \\
 \Rightarrow \frac{dv}{dx} &= v \left[-\sin x \cdot \log(\sin x) + \cos x \cdot \frac{1}{\sin x} \cdot \frac{d}{dx} (\sin x) \right] \\
 \Rightarrow \frac{dv}{dx} &= (\sin x)^{\cos x} \left[-\sin x \log \sin x + \frac{\cos x}{\sin x} \cos x \right] \\
 \Rightarrow \frac{dv}{dx} &= (\sin x)^{\cos x} [-\sin x \log \sin x + \cot x \cos x] \\
 \Rightarrow \frac{dv}{dx} &= (\sin x)^{\cos x} [\cot x \cos x - \sin x \log \sin x] \quad \dots(3)
 \end{aligned}$$

From (1), (2), and (3), we obtain

$$\frac{dy}{dx} = x^{\sin x} \left(\cos x \log x + \frac{\sin x}{x} \right) + (\sin x)^{\cos x} [\cos x \cot x - \sin x \log \sin x]$$

Question 10:

Differentiate the function with respect to x .

$$x^{x \cos x} + \frac{x^2 + 1}{x^2 - 1}$$

Answer

$$\text{Let } y = x^{x \cos x} + \frac{x^2 + 1}{x^2 - 1}$$

$$\text{Also, let } u = x^{x \cos x} \text{ and } v = \frac{x^2 + 1}{x^2 - 1}$$

$$\therefore y = u + v$$

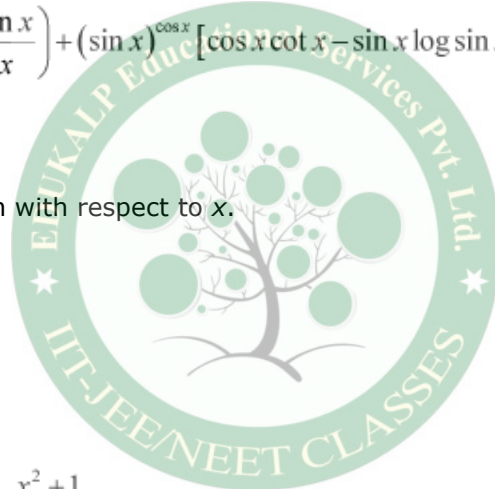
$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \dots(1)$$

$$u = x^{x \cos x}$$

$$\Rightarrow \log u = \log(x^{x \cos x})$$

$$\Rightarrow \log u = x \cos x \log x$$

Differentiating both sides with respect to x , we obtain



$$\begin{aligned}
 \frac{1}{u} \frac{du}{dx} &= \frac{d}{dx}(x) \cdot \cos x \cdot \log x + x \cdot \frac{d}{dx}(\cos x) \cdot \log x + x \cos x \cdot \frac{d}{dx}(\log x) \\
 \Rightarrow \frac{du}{dx} &= u \left[1 \cdot \cos x \cdot \log x + x \cdot (-\sin x) \log x + x \cos x \cdot \frac{1}{x} \right] \\
 \Rightarrow \frac{du}{dx} &= x^{\cos x} (\cos x \log x - x \sin x \log x + \cos x) \\
 \Rightarrow \frac{du}{dx} &= x^{\cos x} [\cos x (1 + \log x) - x \sin x \log x] \quad \dots(2)
 \end{aligned}$$

$$v = \frac{x^2 + 1}{x^2 - 1}$$

$$\Rightarrow \log v = \log(x^2 + 1) - \log(x^2 - 1)$$

Differentiating both sides with respect to x , we obtain

$$\begin{aligned}
 \frac{1}{v} \frac{dv}{dx} &= \frac{2x}{x^2 + 1} - \frac{2x}{x^2 - 1} \\
 \Rightarrow \frac{dv}{dx} &= v \left[\frac{2x(x^2 - 1) - 2x(x^2 + 1)}{(x^2 + 1)(x^2 - 1)} \right] \\
 \Rightarrow \frac{dv}{dx} &= \frac{x^2 + 1}{x^2 - 1} \times \left[\frac{-4x}{(x^2 + 1)(x^2 - 1)} \right] \\
 \Rightarrow \frac{dv}{dx} &= \frac{-4x}{(x^2 - 1)^2} \quad \dots(3)
 \end{aligned}$$

From (1), (2), and (3), we obtain

$$\frac{dy}{dx} = x^{\cos x} [\cos x (1 + \log x) - x \sin x \log x] - \frac{4x}{(x^2 - 1)^2}$$

Question 11:

Differentiate the function with respect to x .

$$(x \cos x)^x + (x \sin x)^{\frac{1}{x}}$$

Answer

$$\text{Let } y = (x \cos x)^x + (x \sin x)^{\frac{1}{x}}$$

$$\text{Also, let } u = (x \cos x)^x \text{ and } v = (x \sin x)^{\frac{1}{x}}$$

$$\therefore y = u + v$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \dots(1)$$

$$u = (x \cos x)^x$$

$$\Rightarrow \log u = \log (x \cos x)^x$$

$$\Rightarrow \log u = x \log (x \cos x)$$

$$\Rightarrow \log u = x [\log x + \log \cos x]$$

$$\Rightarrow \log u = x \log x + x \log \cos x$$

Differentiating both sides with respect to x , we obtain

$$\begin{aligned} \frac{1}{u} \frac{du}{dx} &= \frac{d}{dx} (x \log x) + \frac{d}{dx} (x \log \cos x) \\ \Rightarrow \frac{du}{dx} &= u \left[\left\{ \log x \cdot \frac{d}{dx} (x) + x \cdot \frac{d}{dx} (\log x) \right\} + \left\{ \log \cos x \cdot \frac{d}{dx} (x) + x \cdot \frac{d}{dx} (\log \cos x) \right\} \right] \\ \Rightarrow \frac{du}{dx} &= (x \cos x)^x \left[\left(\log x \cdot 1 + x \cdot \frac{1}{x} \right) + \left\{ \log \cos x \cdot 1 + x \cdot \frac{1}{\cos x} \cdot \frac{d}{dx} (\cos x) \right\} \right] \\ \Rightarrow \frac{du}{dx} &= (x \cos x)^x \left[(\log x + 1) + \left\{ \log \cos x + \frac{x}{\cos x} \cdot (-\sin x) \right\} \right] \\ \Rightarrow \frac{du}{dx} &= (x \cos x)^x \left[(1 + \log x) + (\log \cos x - x \tan x) \right] \\ \Rightarrow \frac{du}{dx} &= (x \cos x)^x \left[1 - x \tan x + (\log x + \log \cos x) \right] \\ \Rightarrow \frac{du}{dx} &= (x \cos x)^x \left[1 - x \tan x + \log (x \cos x) \right] \quad \dots(2) \end{aligned}$$

$$v = (x \sin x)^{\frac{1}{x}}$$

$$\Rightarrow \log v = \log (x \sin x)^{\frac{1}{x}}$$

$$\Rightarrow \log v = \frac{1}{x} \log (x \sin x)$$

$$\Rightarrow \log v = \frac{1}{x} (\log x + \log \sin x)$$

$$\Rightarrow \log v = \frac{1}{x} \log x + \frac{1}{x} \log \sin x$$

Differentiating both sides with respect to x , we obtain

$$\begin{aligned} \frac{1}{v} \frac{dv}{dx} &= \frac{d}{dx} \left(\frac{1}{x} \log x \right) + \frac{d}{dx} \left[\frac{1}{x} \log (\sin x) \right] \\ \Rightarrow \frac{1}{v} \frac{dv}{dx} &= \left[\log x \cdot \frac{d}{dx} \left(\frac{1}{x} \right) + \frac{1}{x} \cdot \frac{d}{dx} (\log x) \right] + \left[\log (\sin x) \cdot \frac{d}{dx} \left(\frac{1}{x} \right) + \frac{1}{x} \cdot \frac{d}{dx} \{ \log (\sin x) \} \right] \\ \Rightarrow \frac{1}{v} \frac{dv}{dx} &= \left[\log x \cdot \left(-\frac{1}{x^2} \right) + \frac{1}{x} \cdot \frac{1}{x} \right] + \left[\log (\sin x) \cdot \left(-\frac{1}{x^2} \right) + \frac{1}{x} \cdot \frac{1}{\sin x} \cdot \frac{d}{dx} (\sin x) \right] \\ \Rightarrow \frac{1}{v} \frac{dv}{dx} &= \frac{1}{x^2} (1 - \log x) + \left[-\frac{\log (\sin x)}{x^2} + \frac{1}{x \sin x} \cdot \cos x \right] \\ \Rightarrow \frac{dv}{dx} &= (x \sin x)^{\frac{1}{x}} \left[\frac{1 - \log x}{x^2} + \frac{-\log (\sin x) + x \cot x}{x^2} \right] \\ \Rightarrow \frac{dv}{dx} &= (x \sin x)^{\frac{1}{x}} \left[\frac{1 - \log x - \log (\sin x) + x \cot x}{x^2} \right] \\ \Rightarrow \frac{dv}{dx} &= (x \sin x)^{\frac{1}{x}} \left[\frac{1 - \log (x \sin x) + x \cot x}{x^2} \right] \quad \dots (3) \end{aligned}$$

From (1), (2), and (3), we obtain

$$\frac{dy}{dx} = (x \cos x)^x [1 - x \tan x + \log (x \cos x)] + (x \sin x)^{\frac{1}{x}} \left[\frac{x \cot x + 1 - \log (x \sin x)}{x^2} \right]$$

Question 12:

Find $\frac{dy}{dx}$ of function.

$$x^y + y^x = 1$$

Answer

The given function is $x^y + y^x = 1$

Let $x^y = u$ and $y^x = v$

Then, the function becomes $u + v = 1$

$$\therefore \frac{du}{dx} + \frac{dv}{dx} = 0 \quad \dots(1)$$

$$u = x^y$$

$$\Rightarrow \log u = \log(x^y)$$

$$\Rightarrow \log u = y \log x$$

Differentiating both sides with respect to x , we obtain

$$\frac{1}{u} \frac{du}{dx} = \log x \frac{dy}{dx} + y \cdot \frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx} = u \left[\log x \frac{dy}{dx} + y \cdot \frac{1}{x} \right]$$

$$\Rightarrow \frac{du}{dx} = x^y \left(\log x \frac{dy}{dx} + \frac{y}{x} \right) \quad \dots(2)$$

$$v = y^x$$

$$\Rightarrow \log v = \log(y^x)$$

$$\Rightarrow \log v = x \log y$$

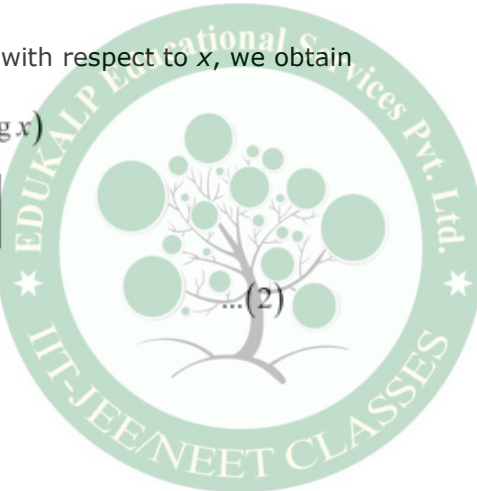
Differentiating both sides with respect to x , we obtain

$$\frac{1}{v} \cdot \frac{dv}{dx} = \log y \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log y)$$

$$\Rightarrow \frac{dv}{dx} = v \left(\log y \cdot 1 + x \cdot \frac{1}{y} \cdot \frac{dy}{dx} \right)$$

$$\Rightarrow \frac{dv}{dx} = y^x \left(\log y + \frac{x}{y} \frac{dy}{dx} \right) \quad \dots(3)$$

From (1), (2), and (3), we obtain



$$x^y \left(\log x \frac{dy}{dx} + \frac{y}{x} \right) + y^x \left(\log y + \frac{x}{y} \frac{dy}{dx} \right) = 0$$

$$\Rightarrow (x^y \log x + xy^{x-1}) \frac{dy}{dx} = -(yx^{y-1} + y^x \log y)$$

$$\therefore \frac{dy}{dx} = -\frac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}$$

Question 13:

Find $\frac{dy}{dx}$ of function.

$$y^x = x^y$$

Answer

The given function is $y^x = x^y$

Taking logarithm on both the sides, we obtain

$$x \log y = y \log x$$

Differentiating both sides with respect to x , we obtain

$$\log y \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log y) = \log x \cdot \frac{d}{dx}(y) + y \cdot \frac{d}{dx}(\log x)$$

$$\Rightarrow \log y \cdot 1 + x \cdot \frac{1}{y} \cdot \frac{dy}{dx} = \log x \cdot \frac{dy}{dx} + y \cdot \frac{1}{x}$$

$$\Rightarrow \log y + \frac{x}{y} \frac{dy}{dx} = \log x \frac{dy}{dx} + \frac{y}{x}$$

$$\Rightarrow \left(\frac{x}{y} - \log x \right) \frac{dy}{dx} = \frac{y}{x} - \log y$$

$$\Rightarrow \left(\frac{x - y \log x}{y} \right) \frac{dy}{dx} = \frac{y - x \log y}{x}$$

$$\therefore \frac{dy}{dx} = \frac{y}{x} \left(\frac{y - x \log y}{x - y \log x} \right)$$

Question 14:

Find $\frac{dy}{dx}$ of function.

$$(\cos x)^y = (\cos y)^x$$

Answer

The given function is $(\cos x)^y = (\cos y)^x$

Taking logarithm on both the sides, we obtain

$$y \log \cos x = x \log \cos y$$

Differentiating both sides, we obtain

$$\begin{aligned} \log \cos x \cdot \frac{dy}{dx} + y \cdot \frac{d}{dx}(\log \cos x) &= \log \cos y \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log \cos y) \\ \Rightarrow \log \cos x \frac{dy}{dx} + y \cdot \frac{1}{\cos x} \cdot \frac{d}{dx}(\cos x) &= \log \cos y \cdot 1 + x \cdot \frac{1}{\cos y} \cdot \frac{d}{dx}(\cos y) \\ \Rightarrow \log \cos x \frac{dy}{dx} + \frac{y}{\cos x} \cdot (-\sin x) &= \log \cos y + \frac{x}{\cos y} \cdot (-\sin y) \cdot \frac{dy}{dx} \\ \Rightarrow \log \cos x \frac{dy}{dx} - y \tan x &= \log \cos y - x \tan y \frac{dy}{dx} \\ \Rightarrow (\log \cos x + x \tan y) \frac{dy}{dx} &= y \tan x + \log \cos y \\ \therefore \frac{dy}{dx} &= \frac{y \tan x + \log \cos y}{x \tan y + \log \cos x} \end{aligned}$$

Question 15:

Find $\frac{dy}{dx}$ of function.

$$xy = e^{(x-y)}$$

Answer

The given function is $xy = e^{(x-y)}$

Taking logarithm on both the sides, we obtain

$$\begin{aligned} \log(xy) &= \log(e^{x-y}) \\ \Rightarrow \log x + \log y &= (x-y) \log e \\ \Rightarrow \log x + \log y &= (x-y) \times 1 \\ \Rightarrow \log x + \log y &= x - y \end{aligned}$$

Differentiating both sides with respect to x , we obtain

$$\frac{d}{dx}(\log x) + \frac{d}{dx}(\log y) = \frac{d}{dx}(x) - \frac{dy}{dx}$$

$$\Rightarrow \frac{1}{x} + \frac{1}{y} \frac{dy}{dx} = 1 - \frac{dy}{dx}$$

$$\Rightarrow \left(1 + \frac{1}{y}\right) \frac{dy}{dx} = 1 - \frac{1}{x}$$

$$\Rightarrow \left(\frac{y+1}{y}\right) \frac{dy}{dx} = \frac{x-1}{x}$$

$$\therefore \frac{dy}{dx} = \frac{y(x-1)}{x(y+1)}$$

Question 16:

Find the derivative of the function given by $f(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)$ and hence find $f'(1)$.

Answer

The given relationship is $f(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)$

Taking logarithm on both the sides, we obtain

$$\log f(x) = \log(1+x) + \log(1+x^2) + \log(1+x^4) + \log(1+x^8)$$

Differentiating both sides with respect to x , we obtain

$$\begin{aligned}\frac{1}{f(x)} \cdot \frac{d}{dx}[f(x)] &= \frac{d}{dx} \log(1+x) + \frac{d}{dx} \log(1+x^2) + \frac{d}{dx} \log(1+x^4) + \frac{d}{dx} \log(1+x^8) \\ \Rightarrow \frac{1}{f(x)} \cdot f'(x) &= \frac{1}{1+x} \cdot \frac{d}{dx}(1+x) + \frac{1}{1+x^2} \cdot \frac{d}{dx}(1+x^2) + \frac{1}{1+x^4} \cdot \frac{d}{dx}(1+x^4) + \frac{1}{1+x^8} \cdot \frac{d}{dx}(1+x^8) \\ \Rightarrow f'(x) &= f(x) \left[\frac{1}{1+x} + \frac{1}{1+x^2} \cdot 2x + \frac{1}{1+x^4} \cdot 4x^3 + \frac{1}{1+x^8} \cdot 8x^7 \right] \\ \therefore f'(x) &= (1+x)(1+x^2)(1+x^4)(1+x^8) \left[\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8} \right] \\ \text{Hence, } f'(1) &= (1+1)(1+1^2)(1+1^4)(1+1^8) \left[\frac{1}{1+1} + \frac{2 \times 1}{1+1^2} + \frac{4 \times 1^3}{1+1^4} + \frac{8 \times 1^7}{1+1^8} \right] \\ &= 2 \times 2 \times 2 \times 2 \left[\frac{1}{2} + \frac{2}{2} + \frac{4}{2} + \frac{8}{2} \right] \\ &= 16 \times \left(\frac{1+2+4+8}{2} \right) \\ &= 16 \times \frac{15}{2} = 120\end{aligned}$$

Question 17:

Differentiate $(x^5 - 5x + 8)(x^3 + 7x + 9)$ in three ways mentioned below

- (i) By using product rule.
- (ii) By expanding the product to obtain a single polynomial.
- (iii) By logarithmic differentiation.

Do they all give the same answer?

Answer

Let $y = (x^5 - 5x + 8)(x^3 + 7x + 9)$

(i)

Let $x^2 - 5x + 8 = u$ and $x^3 + 7x + 9 = v$

$\therefore y = uv$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx} \quad (\text{By using product rule})$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(x^2 - 5x + 8) \cdot (x^3 + 7x + 9) + (x^2 - 5x + 8) \cdot \frac{d}{dx}(x^3 + 7x + 9)$$

$$\Rightarrow \frac{dy}{dx} = (2x - 5)(x^3 + 7x + 9) + (x^2 - 5x + 8)(3x^2 + 7)$$

$$\Rightarrow \frac{dy}{dx} = 2x(x^3 + 7x + 9) - 5(x^3 + 7x + 9) + x^2(3x^2 + 7) - 5x(3x^2 + 7) + 8(3x^2 + 7)$$

$$\Rightarrow \frac{dy}{dx} = (2x^4 + 14x^2 + 18x) - 5x^3 - 35x - 45 + (3x^4 + 7x^2) - 15x^3 - 35x + 24x^2 + 56$$

$$\therefore \frac{dy}{dx} = 5x^4 - 20x^3 + 45x^2 - 52x + 11$$

(ii)

$$y = (x^2 - 5x + 8)(x^3 + 7x + 9)$$

$$= x^2(x^3 + 7x + 9) - 5x(x^3 + 7x + 9) + 8(x^3 + 7x + 9)$$

$$= x^5 + 7x^3 + 9x^2 - 5x^4 - 35x^2 - 45x + 8x^3 + 56x + 72$$

$$= x^5 - 5x^4 + 15x^3 - 26x^2 + 11x + 72$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(x^5 - 5x^4 + 15x^3 - 26x^2 + 11x + 72)$$

$$= \frac{d}{dx}(x^5) - 5 \frac{d}{dx}(x^4) + 15 \frac{d}{dx}(x^3) - 26 \frac{d}{dx}(x^2) + 11 \frac{d}{dx}(x) + \frac{d}{dx}(72)$$

$$= 5x^4 - 5 \times 4x^3 + 15 \times 3x^2 - 26 \times 2x + 11 \times 1 + 0$$

$$= 5x^4 - 20x^3 + 45x^2 - 52x + 11$$

$$(iii) \quad y = (x^2 - 5x + 8)(x^3 + 7x + 9)$$

Taking logarithm on both the sides, we obtain

$$\log y = \log(x^2 - 5x + 8) + \log(x^3 + 7x + 9)$$

Differentiating both sides with respect to x , we obtain

$$\begin{aligned}
 \frac{1}{y} \frac{dy}{dx} &= \frac{d}{dx} \log(x^2 - 5x + 8) + \frac{d}{dx} \log(x^3 + 7x + 9) \\
 \Rightarrow \frac{1}{y} \frac{dy}{dx} &= \frac{1}{x^2 - 5x + 8} \cdot \frac{d}{dx}(x^2 - 5x + 8) + \frac{1}{x^3 + 7x + 9} \cdot \frac{d}{dx}(x^3 + 7x + 9) \\
 \Rightarrow \frac{dy}{dx} &= y \left[\frac{1}{x^2 - 5x + 8} \times (2x - 5) + \frac{1}{x^3 + 7x + 9} \times (3x^2 + 7) \right] \\
 \Rightarrow \frac{dy}{dx} &= (x^2 - 5x + 8)(x^3 + 7x + 9) \left[\frac{2x - 5}{x^2 - 5x + 8} + \frac{3x^2 + 7}{x^3 + 7x + 9} \right] \\
 \Rightarrow \frac{dy}{dx} &= (x^2 - 5x + 8)(x^3 + 7x + 9) \left[\frac{(2x - 5)(x^3 + 7x + 9) + (3x^2 + 7)(x^2 - 5x + 8)}{(x^2 - 5x + 8)(x^3 + 7x + 9)} \right] \\
 \Rightarrow \frac{dy}{dx} &= 2x(x^3 + 7x + 9) - 5(x^3 + 7x + 9) + 3x^2(x^2 - 5x + 8) + 7(x^2 - 5x + 8) \\
 \Rightarrow \frac{dy}{dx} &= (2x^4 + 14x^2 + 18x) - 5x^3 - 35x - 45 + (3x^4 - 15x^3 + 24x^2) + (7x^2 - 35x + 56) \\
 \Rightarrow \frac{dy}{dx} &= 5x^4 - 20x^3 + 45x^2 - 52x + 11
 \end{aligned}$$

From the above three observations, it can be concluded that all the results of $\frac{dy}{dx}$ are same.

Question 18:

If u , v and w are functions of x , then show that

$$\frac{d}{dx}(u.v.w) = \frac{du}{dx}v.w + u.\frac{dv}{dx}.w + u.v.\frac{dw}{dx}$$

in two ways-first by repeated application of product rule, second by logarithmic differentiation.

Answer

Let $y = u.v.w = u.(v.w)$

By applying product rule, we obtain

$$\begin{aligned}\frac{dy}{dx} &= \frac{du}{dx} \cdot (v \cdot w) + u \cdot \frac{d}{dx}(v \cdot w) \\ \Rightarrow \frac{dy}{dx} &= \frac{du}{dx} \cdot v \cdot w + u \left[\frac{dv}{dx} \cdot w + v \cdot \frac{dw}{dx} \right] \quad (\text{Again applying product rule}) \\ \Rightarrow \frac{dy}{dx} &= \frac{du}{dx} \cdot v \cdot w + u \cdot \frac{dv}{dx} \cdot w + u \cdot v \cdot \frac{dw}{dx}\end{aligned}$$

By taking logarithm on both sides of the equation $y = u \cdot v \cdot w$, we obtain

$$\log y = \log u + \log v + \log w$$

Differentiating both sides with respect to x , we obtain

$$\begin{aligned}\frac{1}{y} \cdot \frac{dy}{dx} &= \frac{d}{dx}(\log u) + \frac{d}{dx}(\log v) + \frac{d}{dx}(\log w) \\ \Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} &= \frac{1}{u} \frac{du}{dx} + \frac{1}{v} \frac{dv}{dx} + \frac{1}{w} \frac{dw}{dx} \\ \Rightarrow \frac{dy}{dx} &= y \left(\frac{1}{u} \frac{du}{dx} + \frac{1}{v} \frac{dv}{dx} + \frac{1}{w} \frac{dw}{dx} \right) \\ \Rightarrow \frac{dy}{dx} &= u \cdot v \cdot w \cdot \left(\frac{1}{u} \frac{du}{dx} + \frac{1}{v} \frac{dv}{dx} + \frac{1}{w} \frac{dw}{dx} \right) \\ \therefore \frac{dy}{dx} &= \frac{du}{dx} \cdot v \cdot w + u \cdot \frac{dv}{dx} \cdot w + u \cdot v \cdot \frac{dw}{dx}\end{aligned}$$

