Exercise 5.7

Question 1:

Find the second order derivatives of the function.

x^2+3x+2

Answer

Let $y=x^2+3x+2$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}(3x)+\frac{d}{dx}(2)=2x+3+0=2x+3$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}(2x+3)=\frac{d}{dx}(2x)+\frac{d}{dx}(3)=2+0=2$$

Question 2:

Find the second order derivatives of the function.

x^{20}

Answer

Let $y=x^{20}$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left(x^{20}\right)=20x^{19}$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left(20x^{19}\right)=20\frac{d}{dx}\left(x^{19}\right)=20\cdot19\cdot x^{18}=380x^{18}$$

Question 3:

Find the second order derivatives of the function.

$x\cdot\cos x$

Answer

Let $y=x\cdot\cos x$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}(x\cdot\cos x)=\cos x\cdot\frac{d}{dx}(x)+x\frac{d}{dx}(\cos x)=\cos x\cdot 1+x(-\sin x)=\cos x-x\sin x$$

$$\therefore\frac{d^2y}{dx^2}=\frac{d}{dx}[\cos x-x\sin x]=\frac{d}{dx}(\cos x)-\frac{d}{dx}(x\sin x)$$

$$=-\sin x-\left[\sin x\cdot\frac{d}{dx}(x)+x\cdot\frac{d}{dx}(\sin x)\right]$$

$$=-\sin x-(\sin x+x\cos x)$$

$$=-(x\cos x+2\sin x)$$

Question 4:

Find the second order derivatives of the function.

$\log x$

Answer

Let $y=\log x$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}(\log x)=\frac{1}{x}$$

$$\therefore\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1}{x}\right)=\frac{-1}{x^2}$$

Question 5:

Find the second order derivatives of the function.

$x^3\log x$

Answer

Let $y=x^3\log x$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left[x^3\log x\right]=\log x\cdot\frac{d}{dx}\left(x^3\right)+x^3\cdot\frac{d}{dx}(\log x)$$
$$=\log x\cdot 3x^2+x^3\cdot\frac{1}{x}=\log x\cdot 3x^2+x^2$$
$$=x^2(1+3\log x)$$
$$\therefore\frac{d^2y}{dx^2}=\frac{d}{dx}\left[x^2(1+3\log x)\right]$$
$$=(1+3\log x)\cdot\frac{d}{dx}\left(x^2\right)+x^2\frac{d}{dx}(1+3\log x)$$
$$=(1+3\log x)\cdot 2x+x^2\cdot\frac{3}{x}$$
$$=2x+6x\log x+3x$$
$$=5x+6x\log x$$
$$=x(5+6\log x)$$

Question 6:

Find the second order derivatives of the function.

$e^x\sin 5x$

Answer

Let $y=e^x\sin 5x$

$$\frac{dy}{dx}=\frac{d}{dx}\left(e^x\sin 5x\right)=\sin 5x\cdot\frac{d}{dx}\left(e^x\right)+e^x\frac{d}{dx}(\sin 5x)$$
$$=\sin 5x\cdot e^x+e^x\cdot\cos 5x\cdot\frac{d}{dx}(5x)=e^x\sin 5x+e^x\cos 5x\cdot 5$$
$$=e^x(\sin 5x+5\cos 5x)$$
$$\therefore\frac{d^2y}{dx^2}=\frac{d}{dx}\left[e^x(\sin 5x+5\cos 5x)\right]$$
$$=(\sin 5x+5\cos 5x)\cdot\frac{d}{dx}\left(e^x\right)+e^x\cdot\frac{d}{dx}(\sin 5x+5\cos 5x)$$
$$=(\sin 5x+5\cos 5x)e^x+e^x\left[\cos 5x\cdot\frac{d}{dx}(5x)+5(-\sin 5x)\cdot\frac{d}{dx}(5x)\right]$$
$$=e^x(\sin 5x+5\cos 5x)+e^x(5\cos 5x-25\sin 5x)$$

Then,
$$=e^x(10\cos 5x-24\sin 5x)=2e^x(5\cos 5x-12\sin 5x)$$

Question 7:

Find the second order derivatives of the function.

$e^{6x}\cos 3x$

Answer

Let $y=e^{6x}\cos 3x$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left(e^{6x}\cdot\cos 3x\right)=\cos 3x\cdot\frac{d}{dx}\left(e^{6x}\right)+e^{6x}\cdot\frac{d}{dx}\left(\cos 3x\right)$$

$$=\cos 3x\cdot e^{6x}\cdot\frac{d}{dx}(6x)+e^{6x}\cdot(-\sin 3x)\cdot\frac{d}{dx}(3x)$$

$$=6e^{6x}\cos 3x-3e^{6x}\sin 3x \qquad (1)$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left(6e^{6x}\cos 3x-3e^{6x}\sin 3x\right)=6\cdot\frac{d}{dx}\left(e^{6x}\cos 3x\right)-3\cdot\frac{d}{dx}\left(e^{6x}\sin 3x\right)$$

$$=6\cdot\left[6e^{6x}\cos 3x-3e^{6x}\sin 3x\right]-3\cdot\left[\sin 3x\cdot\frac{d}{dx}\left(e^{6x}\right)+e^{6x}\cdot\frac{d}{dx}(\sin 3x)\right] \qquad [\text{Using (1)}]$$

$$=36e^{6x}\cos 3x-18e^{6x}\sin 3x-3\left[\sin 3x\cdot e^{6x}\cdot 6+e^{6x}\cdot\cos 3x\cdot 3\right]$$

$$=36e^{6x}\cos 3x-18e^{6x}\sin 3x-18e^{6x}\sin 3x-9e^{6x}\cos 3x$$

$$=27e^{6x}\cos 3x-36e^{6x}\sin 3x$$

$$=9e^{6x}\left(3\cos 3x-4\sin 3x\right)$$

Question 8:

Find the second order derivatives of the function.

$\tan^{-1}x$

Answer

Let $y=\tan^{-1}x$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left(\tan^{-1}x\right)=\frac{1}{1+x^2}$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1}{1+x^2}\right)=\frac{d}{dx}\left(1+x^2\right)^{-1}=(-1)\cdot\left(1+x^2\right)^{-2}\cdot\frac{d}{dx}\left(1+x^2\right)$$

$$=\frac{-1}{\left(1+x^2\right)^2}\times 2x=\frac{-2x}{\left(1+x^2\right)^2}$$

Question 9:

Find the second order derivatives of the function.

$\log(\log x)$

Answer

Let $y=\log(\log x)$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left[\log(\log x)\right]=\frac{1}{\log x}\cdot\frac{d}{dx}(\log x)=\frac{1}{x\log x}=(x\log x)^{-1}$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left[(x\log x)^{-1}\right]=(-1)\cdot(x\log x)^{-2}\cdot\frac{d}{dx}(x\log x)$$

$$=\frac{-1}{(x\log x)^2}\cdot\left[\log x\cdot\frac{d}{dx}(x)+x\cdot\frac{d}{dx}(\log x)\right]$$

$$=\frac{-1}{(x\log x)^2}\cdot\left[\log x\cdot 1+x\cdot\frac{1}{x}\right]=\frac{-(1+\log x)}{(x\log x)^2}$$

Question 10:

Find the second order derivatives of the function.

$\sin(\log x)$

Answer

Let $y=\sin(\log x)$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left[\sin(\log x)\right] = \cos(\log x)\cdot\frac{d}{dx}(\log x) = \frac{\cos(\log x)}{x}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{\cos(\log x)}{x}\right]$$

$$= \frac{x\cdot\frac{d}{dx}\left[\cos(\log x)\right] - \cos(\log x)\cdot\frac{d}{dx}(x)}{x^2}$$

$$= \frac{x\cdot\left[-\sin(\log x)\cdot\frac{d}{dx}(\log x)\right] - \cos(\log x).1}{x^2}$$

$$= \frac{-x\sin(\log x).\frac{1}{x} - \cos(\log x)}{x^2}$$

$$= \frac{-\left[\sin(\log x) + \cos(\log x)\right]}{x^2}$$

Question 11:

If $y = 5\cos x - 3\sin x$, prove that $\frac{d^2y}{dx^2} + y = 0$

Answer

It is given that, $y = 5\cos x - 3\sin x$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}(5\cos x)-\frac{d}{dx}(3\sin x)=5\frac{d}{dx}(\cos x)-3\frac{d}{dx}(\sin x)$$
$$=5(-\sin x)-3\cos x=-(5\sin x+3\cos x)$$
$$\therefore\frac{d^2y}{dx^2}=\frac{d}{dx}\left[-(5\sin x+3\cos x)\right]$$
$$=-\left[5\cdot\frac{d}{dx}(\sin x)+3\cdot\frac{d}{dx}(\cos x)\right]$$
$$=-\left[5\cos x+3(-\sin x)\right]$$
$$=-\left[5\cos x-3\sin x\right]$$
$$=-y$$
$$\therefore\frac{d^2y}{dx^2}+y=0$$

Hence, proved.

Question 12:

If $y=\cos^{-1}x$, find $\frac{d^2y}{dx^2}$ in terms of y alone.

Answer

It is given that, $y=\cos^{-1}x$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left(\cos^{-1}x\right)=\frac{-1}{\sqrt{1-x^2}}=-\left(1-x^2\right)^{\frac{-1}{2}}$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left[-\left(1-x^2\right)^{\frac{-1}{2}}\right]$$

$$=-\left(-\frac{1}{2}\right)\cdot\left(1-x^2\right)^{\frac{-3}{2}}\cdot\frac{d}{dx}\left(1-x^2\right)$$

$$=\frac{1}{2\sqrt{\left(1-x^2\right)^3}}\times(-2x)$$

$$\Rightarrow\frac{d^2y}{dx^2}=\frac{-x}{\sqrt{\left(1-x^2\right)^3}}\qquad\ldots(\mathrm{i})$$

$y=\cos^{-1}x\Rightarrow x=\cos y$

Putting $x=\cos y$ in equation (i), we obtain

$$\frac{d^2y}{dx^2}=\frac{-\cos y}{\sqrt{\left(1-\cos^2 y\right)^3}}$$

$$\Rightarrow\frac{d^2y}{dx^2}=\frac{-\cos y}{\sqrt{\left(\sin^2 y\right)^3}}$$

$$=\frac{-\cos y}{\sin^3 y}$$

$$=\frac{-\cos y}{\sin y}\times\frac{1}{\sin^2 y}$$

$$\Rightarrow\frac{d^2y}{dx^2}=-\cot y\cdot\operatorname{cosec}^2 y$$

Question 13:

If $y=3\cos(\log x)+4\sin(\log x)$, show that $x^2y_2+xy_1+y=0$

Answer

It is given that, $y=3\cos(\log x)+4\sin(\log x)$

Then,

$$y_1 = 3\cdot\frac{d}{dx}\left[\cos(\log x)\right]+4\cdot\frac{d}{dx}\left[\sin(\log x)\right]$$

$$=3\cdot\left[-\sin(\log x)\cdot\frac{d}{dx}(\log x)\right]+4\cdot\left[\cos(\log x)\cdot\frac{d}{dx}(\log x)\right]$$

$$\therefore y_1=\frac{-3\sin(\log x)}{x}+\frac{4\cos(\log x)}{x}=\frac{4\cos(\log x)-3\sin(\log x)}{x}$$

$$\therefore y_2=\frac{d}{dx}\left(\frac{4\cos(\log x)-3\sin(\log x)}{x}\right)$$

$$=\frac{x\{4\cos(\log x)-3\sin(\log x)\}'-\{4\cos(\log x)-3\sin(\log x)\}(x)'}{x^2}$$

$$=\frac{x\left[4\{\cos(\log x)\}'-3\{\sin(\log x)\}'\right]-\{4\cos(\log x)-3\sin(\log x)\}.1}{x^2}$$

$$=\frac{x\left[-4\sin(\log x).(\log x)'-3\cos(\log x).(\log x)'\right]-4\cos(\log x)+3\sin(\log x)}{x^2}$$

$$=\frac{x\left[-4\sin(\log x).\frac{1}{x}-3\cos(\log x).\frac{1}{x}\right]-4\cos(\log x)+3\sin(\log x)}{x^2}$$

$$=\frac{-4\sin(\log x)-3\cos(\log x)-4\cos(\log x)+3\sin(\log x)}{x^2}$$

$$=\frac{-\sin(\log x)-7\cos(\log x)}{x^2}$$

$$\therefore x^2y_2+xy_1+y$$

$$=x^2\left(\frac{-\sin(\log x)-7\cos(\log x)}{x^2}\right)+x\left(\frac{4\cos(\log x)-3\sin(\log x)}{x}\right)+3\cos(\log x)+4\sin(\log x)$$

$$=-\sin(\log x)-7\cos(\log x)+4\cos(\log x)-3\sin(\log x)+3\cos(\log x)+4\sin(\log x)$$

$$=0$$

Hence, proved.

Question 14:

If $y=Ae^{mx}+Be^{nx}$, show that $\frac{d^2y}{dx^2}-(m+n)\frac{dy}{dx}+mny=0$

Answer

It is given that, $y = Ae^{mx} + Be^{nx}$

Then,

$$\frac{dy}{dx} = A\cdot\frac{d}{dx}\left(e^{mx}\right) + B\cdot\frac{d}{dx}\left(e^{nx}\right) = A\cdot e^{mx}.\frac{d}{dx}(mx) + B\cdot e^{nx}\cdot\frac{d}{dx}(nx) = Ame^{mx} + Bne^{nx}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(Ame^{mx} + Bne^{nx}\right) = Am\cdot\frac{d}{dx}\left(e^{mx}\right) + Bn\cdot\frac{d}{dx}\left(e^{nx}\right)$$

$$= Am\cdot e^{mx}\cdot\frac{d}{dx}(mx) + Bn\cdot e^{nx}\cdot\frac{d}{dx}(nx) = Am^2e^{mx} + Bn^2e^{nx}$$

$$\therefore \frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny$$

$$= Am^2e^{mx} + Bn^2e^{nx} - (m+n)\cdot\left(Ame^{mx} + Bne^{nx}\right) + mn\left(Ae^{mx} + Be^{nx}\right)$$

$$= Am^2e^{mx} + Bn^2e^{nx} - Am^2e^{mx} - Bmne^{nx} - Amne^{mx} - Bn^2e^{nx} + Amne^{mx} + Bmne^{nx}$$

$$= 0$$

Hence, proved.

Question 15:

If $y = 500e^{7x} + 600e^{-7x}$, show that $\frac{d^2y}{dx^2} = 49y$

Answer

It is given that, $y = 500e^{7x} + 600e^{-7x}$

Then,

$$\frac{dy}{dx}=500.\frac{d}{dx}\left(e^{7x}\right)+600.\frac{d}{dx}\left(e^{-7x}\right)$$
$$=500\cdot e^{7x}\cdot\frac{d}{dx}(7x)+600\cdot e^{-7x}\cdot\frac{d}{dx}(-7x)$$
$$=3500e^{7x}-4200e^{-7x}$$
$$\therefore\frac{d^2y}{dx^2}=3500\cdot\frac{d}{dx}\left(e^{7x}\right)-4200.\frac{d}{dx}\left(e^{-7x}\right)$$
$$=3500\cdot e^{7x}.\frac{d}{dx}(7x)-4200\cdot e^{-7x}\cdot\frac{d}{dx}(-7x)$$
$$=7\times3500\cdot e^{7x}+7\times4200\cdot e^{-7x}$$
$$=49\times500e^{7x}+49\times600e^{-7x}$$
$$=49\left(500e^{7x}+600e^{-7x}\right)$$
$$=49y$$

Hence, proved.

Question 16:

If $e^y(x+1)=1$, show that $\frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^2$

Answer

The given relationship is $e^y(x+1)=1$

$$e^y(x+1)=1$$
$$\Rightarrow e^y=\frac{1}{x+1}$$

Taking logarithm on both the sides, we obtain

$$y=\log\frac{1}{(x+1)}$$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{dy}{dx}=(x+1)\frac{d}{dx}\left(\frac{1}{x+1}\right)=(x+1).\frac{-1}{(x+1)^2}=\frac{-1}{x+1}$$

$$\therefore \frac{d^2y}{dx^2}=-\frac{d}{dx}\left(\frac{1}{x+1}\right)=-\left(\frac{-1}{(x+1)^2}\right)=\frac{1}{(x+1)^2}$$

$$\Rightarrow \frac{d^2y}{dx^2}=\left(\frac{-1}{x+1}\right)^2$$

$$\Rightarrow \frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^2$$

Hence, proved.

Question 17:

If $y=\left(\tan^{-1}x\right)^2$, show that $\left(x^2+1\right)^2 y_2+2x\left(x^2+1\right)y_1=2$

Answer

The given relationship is $y=\left(\tan^{-1}x\right)^2$

Then,

$$y_1=2\tan^{-1}x\frac{d}{dx}\left(\tan^{-1}x\right)$$

$$\Rightarrow y_1=2\tan^{-1}x.\frac{1}{1+x^2}$$

$$\Rightarrow \left(1+x^2\right)y_1=2\tan^{-1}x$$

Again differentiating with respect to x on both the sides, we obtain

$$\left(1+x^2\right)y_2+2xy_1=2\left(\frac{1}{1+x^2}\right)$$

$$\Rightarrow \left(1+x^2\right)^2 y_2+2x\left(1+x^2\right)y_1=2$$

Hence, proved.