Miscellaneous Solutions

Question 1:

$\left(3x^2-9x+5\right)^9$

Answer

Let $y=\left(3x^2-9x+5\right)^9$

Using chain rule, we obtain

$$\begin{aligned}\frac{dy}{dx}&=\frac{d}{dx}\left(3x^2-9x+5\right)^9\\&=9\left(3x^2-9x+5\right)^8\cdot\frac{d}{dx}\left(3x^2-9x+5\right)\\&=9\left(3x^2-9x+5\right)^8\cdot\left(6x-9\right)\\&=9\left(3x^2-9x+5\right)^8\cdot3\left(2x-3\right)\\&=27\left(3x^2-9x+5\right)^8\left(2x-3\right)\end{aligned}$$

Question 2:

$\sin^3 x+\cos^6 x$

Answer

Let $y=\sin^3 x+\cos^6 x$

$$\begin{aligned}\therefore\frac{dy}{dx}&=\frac{d}{dx}\left(\sin^3 x\right)+\frac{d}{dx}\left(\cos^6 x\right)\\&=3\sin^2 x\cdot\frac{d}{dx}\left(\sin x\right)+6\cos^5 x\cdot\frac{d}{dx}\left(\cos x\right)\\&=3\sin^2 x\cdot\cos x+6\cos^5 x\cdot\left(-\sin x\right)\\&=3\sin x\cos x\left(\sin x-2\cos^4 x\right)\end{aligned}$$

Question 3:

$\left(5x\right)^{3\cos 2x}$

Answer

Let $y=(5x)^{3\cos 2x}$

Taking logarithm on both the sides, we obtain

$\log y=3\cos 2x\log 5x$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx}=3\left[\log 5x\cdot\frac{d}{dx}(\cos 2x)+\cos 2x\cdot\frac{d}{dx}(\log 5x)\right]$$

$$\Rightarrow\frac{dy}{dx}=3y\left[\log 5x(-\sin 2x)\cdot\frac{d}{dx}(2x)+\cos 2x\cdot\frac{1}{5x}\cdot\frac{d}{dx}(5x)\right]$$

$$\Rightarrow\frac{dy}{dx}=3y\left[-2\sin 2x\log 5x+\frac{\cos 2x}{x}\right]$$

$$\Rightarrow\frac{dy}{dx}=3y\left[\frac{3\cos 2x}{x}-6\sin 2x\log 5x\right]$$

$$\therefore\frac{dy}{dx}=(5x)^{3\cos 2x}\left[\frac{3\cos 2x}{x}-6\sin 2x\log 5x\right]$$

Question 4:

$\sin^{-1}\left(x\sqrt{x}\right),\ 0\le x\le 1$

Answer

Let $y=\sin^{-1}\left(x\sqrt{x}\right)$

Using chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\sin^{-1}\left(x\sqrt{x}\right)$$
$$= \frac{1}{\sqrt{1-\left(x\sqrt{x}\right)^2}} \times \frac{d}{dx}\left(x\sqrt{x}\right)$$
$$= \frac{1}{\sqrt{1-x^3}} \cdot \frac{d}{dx}\left(x^{\frac{3}{2}}\right)$$
$$= \frac{1}{\sqrt{1-x^3}} \times \frac{3}{2} \cdot x^{\frac{1}{2}}$$
$$= \frac{3\sqrt{x}}{2\sqrt{1-x^3}}$$
$$= \frac{3}{2}\sqrt{\frac{x}{1-x^3}}$$

Question 5:

$$\frac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}}, \quad -2 < x < 2$$

Answer

Let $y = \dfrac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}}$

By quotient rule, we obtain

$$\frac{dy}{dx} = \frac{\sqrt{2x+7}\,\frac{d}{dx}\left(\cos^{-1}\frac{x}{2}\right) - \left(\cos^{-1}\frac{x}{2}\right)\frac{d}{dx}\left(\sqrt{2x+7}\right)}{\left(\sqrt{2x+7}\right)^2}$$

$$= \frac{\sqrt{2x+7}\left[\dfrac{-1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\cdot\dfrac{d}{dx}\left(\dfrac{x}{2}\right)\right] - \left(\cos^{-1}\dfrac{x}{2}\right)\dfrac{1}{2\sqrt{2x+7}}\cdot\dfrac{d}{dx}(2x+7)}{2x+7}$$

$$= \frac{\sqrt{2x+7}\,\dfrac{-1}{\sqrt{4-x^2}} - \left(\cos^{-1}\dfrac{x}{2}\right)\dfrac{2}{2\sqrt{2x+7}}}{2x+7}$$

$$= \frac{-\sqrt{2x+7}}{\sqrt{4-x^2}\times(2x+7)} - \frac{\cos^{-1}\frac{x}{2}}{\left(\sqrt{2x+7}\right)(2x+7)}$$

$$= -\left[\frac{1}{\sqrt{4-x^2}\,\sqrt{2x+7}} + \frac{\cos^{-1}\frac{x}{2}}{(2x+7)^{\frac{3}{2}}}\right]$$

Question 6:

$$\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right], 0 < x < \frac{\pi}{2}$$

Answer

Let $y=\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]$ ...(1)

Then, $\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}$

$=\frac{\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)^2}{\left(\sqrt{1+\sin x}-\sqrt{1-\sin x}\right)\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)}$

$=\frac{(1+\sin x)+(1-\sin x)+2\sqrt{(1-\sin x)(1+\sin x)}}{(1+\sin x)-(1-\sin x)}$

$=\frac{2+2\sqrt{1-\sin^2 x}}{2\sin x}$

$=\frac{1+\cos x}{\sin x}$

$=\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}$

$=\cot\frac{x}{2}$

Therefore, equation (1) becomes

$y=\cot^{-1}\left(\cot\frac{x}{2}\right)$

$\Rightarrow y=\frac{x}{2}$

$\therefore \frac{dy}{dx}=\frac{1}{2}\frac{d}{dx}(x)$

$\Rightarrow \frac{dy}{dx}=\frac{1}{2}$

Question 7:

$(\log x)^{\log x},\ x>1$

Answer

Let $y=(\log x)^{\log x}$

Taking logarithm on both the sides, we obtain

$$\log y = \log x \cdot \log(\log x)$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\left[\log x \cdot \log(\log x)\right]$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log(\log x).\frac{d}{dx}(\log x) + \log x \cdot \frac{d}{dx}\left[\log(\log x)\right]$$

$$\Rightarrow \frac{dy}{dx} = y\left[\log(\log x).\frac{1}{x} + \log x \cdot \frac{1}{\log x} \cdot \frac{d}{dx}(\log x)\right]$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{1}{x}\log(\log x) + \frac{1}{x}\right]$$

$$\therefore \frac{dy}{dx} = (\log x)^{\log x}\left[\frac{1}{x} + \frac{\log(\log x)}{x}\right]$$

Question 8:

$\cos(a\cos x + b\sin x)$, for some constant *a* and *b*.

Answer

$$\text{Let } y = \cos(a\cos x + b\sin x)$$

By using chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\cos(a\cos x + b\sin x)$$

$$\Rightarrow \frac{dy}{dx} = -\sin(a\cos x + b\sin x) \cdot \frac{d}{dx}(a\cos x + b\sin x)$$

$$= -\sin(a\cos x + b\sin x) \cdot \left[a(-\sin x) + b\cos x\right]$$

$$= (a\sin x - b\cos x) \cdot \sin(a\cos x + b\sin x)$$

Question 9:

$$(\sin x - \cos x)^{(\sin x - \cos x)}, \quad \frac{\pi}{4} < x < \frac{3\pi}{4}$$

Answer

$\text{Let } y=(\sin x-\cos x)^{(\sin x-\cos x)}$

Taking logarithm on both the sides, we obtain

$\log y=\log\left[(\sin x-\cos x)^{(\sin x-\cos x)}\right]$

$\Rightarrow \log y=(\sin x-\cos x)\cdot\log(\sin x-\cos x)$

Differentiating both sides with respect to *x*, we obtain

$\frac{1}{y}\frac{dy}{dx}=\frac{d}{dx}\left[(\sin x-\cos x)\log(\sin x-\cos x)\right]$

$\Rightarrow \frac{1}{y}\frac{dy}{dx}=\log(\sin x-\cos x)\cdot\frac{d}{dx}(\sin x-\cos x)+(\sin x-\cos x)\cdot\frac{d}{dx}\log(\sin x-\cos x)$

$\Rightarrow \frac{1}{y}\frac{dy}{dx}=\log(\sin x-\cos x)\cdot(\cos x+\sin x)+(\sin x-\cos x)\cdot\frac{1}{(\sin x-\cos x)}\cdot\frac{d}{dx}(\sin x-\cos x)$

$\Rightarrow \frac{dy}{dx}=(\sin x-\cos x)^{(\sin x-\cos x)}\left[(\cos x+\sin x)\cdot\log(\sin x-\cos x)+(\cos x+\sin x)\right]$

$\therefore \frac{dy}{dx}=(\sin x-\cos x)^{(\sin x-\cos x)}(\cos x+\sin x)\left[1+\log(\sin x-\cos x)\right]$

Question 10:

$x^x+x^a+a^x+a^a$, for some fixed $a>0$ and $x>0$

Answer

$\text{Let } y=x^x+x^a+a^x+a^a$

Also, let $x^x=u,\ x^a=v,\ a^x=w,$ and $a^a=s$

$\therefore y=u+v+w+s$

$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}+\frac{dw}{dx}+\frac{ds}{dx} \qquad ...(1)$

$u=x^x$

$\Rightarrow \log u=\log x^x$

$\Rightarrow \log u=x\log x$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{u}\frac{du}{dx} = \log x \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx} = u\left[\log x \cdot 1 + x \cdot \frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx} = x^x\left[\log x + 1\right] = x^x\left(1 + \log x\right) \qquad ...(2)$$

$$v = x^a$$

$$\therefore \frac{dv}{dx} = \frac{d}{dx}\left(x^a\right)$$

$$\Rightarrow \frac{dv}{dx} = ax^{a-1} \qquad ...(3)$$

$$w = a^x$$

$$\Rightarrow \log w = \log a^x$$

$$\Rightarrow \log w = x \log a$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{w} \cdot \frac{dw}{dx} = \log a \cdot \frac{d}{dx}(x)$$

$$\Rightarrow \frac{dw}{dx} = w \log a$$

$$\Rightarrow \frac{dw}{dx} = a^x \log a \qquad ...(4)$$

$s = a^a$

Since a is constant, a^a is also a constant.

$$\therefore \frac{ds}{dx} = 0 \qquad ...(5)$$

From (1), (2), (3), (4), and (5), we obtain

$$\frac{dy}{dx} = x^x\left(1 + \log x\right) + ax^{a-1} + a^x \log a + 0$$

$$= x^x\left(1 + \log x\right) + ax^{a-1} + a^x \log a$$

Question 11:

$x^{x^2-3}+(x-3)^{x^2}$, for $x>3$

Answer

Let $y=x^{x^2-3}+(x-3)^{x^2}$

Also, let $u=x^{x^2-3}$ and $v=(x-3)^{x^2}$

$\therefore y=u+v$

Differentiating both sides with respect to *x,* we obtain

$$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \qquad ...(1)$$

$u=x^{x^2-3}$

$\therefore \log u=\log\left(x^{x^2-3}\right)$

$\log u=\left(x^2-3\right)\log x$

Differentiating with respect to *x*, we obtain

$$\frac{1}{u}\cdot\frac{du}{dx}=\log x\cdot\frac{d}{dx}\left(x^2-3\right)+\left(x^2-3\right)\cdot\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx}=\log x\cdot 2x+\left(x^2-3\right)\cdot\frac{1}{x}$$

$$\Rightarrow \frac{du}{dx}=x^{x^2-3}\cdot\left[\frac{x^2-3}{x}+2x\log x\right]$$

Also,

$v=(x-3)^{x^2}$

$\therefore \log v=\log(x-3)^{x^2}$

$\Rightarrow \log v=x^2\log(x-3)$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{v}\cdot\frac{dv}{dx}=\log(x-3)\cdot\frac{d}{dx}(x^2)+x^2\cdot\frac{d}{dx}[\log(x-3)]$$

$$\Rightarrow\frac{1}{v}\frac{dv}{dx}=\log(x-3)\cdot 2x+x^2\cdot\frac{1}{x-3}\cdot\frac{d}{dx}(x-3)$$

$$\Rightarrow\frac{dv}{dx}=v\left[2x\log(x-3)+\frac{x^2}{x-3}\cdot 1\right]$$

$$\Rightarrow\frac{dv}{dx}=(x-3)^{x^2}\left[\frac{x^2}{x-3}+2x\log(x-3)\right]$$

Substituting the expressions of $\frac{du}{dx}$ and $\frac{dv}{dx}$ in equation (1), we obtain

$$\frac{dy}{dx}=x^{x^2-3}\left[\frac{x^2-3}{x}+2x\log x\right]+(x-3)^{x^2}\left[\frac{x^2}{x-3}+2x\log(x-3)\right]$$

Question 12:

Find $\frac{dy}{dx}$, if $y=12(1-\cos t), x=10(t-\sin t), -\frac{\pi}{2}<t<\frac{\pi}{2}$

Answer

It is given that, $y=12(1-\cos t), x=10(t-\sin t)$

$$\therefore\frac{dx}{dt}=\frac{d}{dt}[10(t-\sin t)]=10\cdot\frac{d}{dt}(t-\sin t)=10(1-\cos t)$$

$$\frac{dy}{dt}=\frac{d}{dt}[12(1-\cos t)]=12\cdot\frac{d}{dt}(1-\cos t)=12\cdot[0-(-\sin t)]=12\sin t$$

$$\therefore\frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{12\sin t}{10(1-\cos t)}=\frac{12\cdot 2\sin\frac{t}{2}\cdot\cos\frac{t}{2}}{10\cdot 2\sin^2\frac{t}{2}}=\frac{6}{5}\cot\frac{t}{2}$$

Question 13:

Find $\frac{dy}{dx}$, if $y=\sin^{-1}x+\sin^{-1}\sqrt{1-x^2},\ -1\le x\le 1$

Answer

It is given that, $y = \sin^{-1} x + \sin^{-1}\sqrt{1-x^2}$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left[\sin^{-1} x + \sin^{-1}\sqrt{1-x^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(\sin^{-1} x\right) + \frac{d}{dx}\left(\sin^{-1}\sqrt{1-x^2}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}} \cdot \frac{d}{dx}\left(\sqrt{1-x^2}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{x} \cdot \frac{1}{2\sqrt{1-x^2}} \cdot \frac{d}{dx}\left(1-x^2\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{2x\sqrt{1-x^2}}(-2x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \frac{dy}{dx} = 0$$

Question 14:

If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, for, $-1 < x < 1$, prove that

$$\frac{dy}{dx} = -\frac{1}{(1+x)^2}$$

Answer

It is given that,

$$x\sqrt{1+y} + y\sqrt{1+x} = 0$$

$\Rightarrow x\sqrt{1+y} = -y\sqrt{1+x}$

Squaring both sides, we obtain

$x^2(1+y) = y^2(1+x)$

$\Rightarrow x^2 + x^2y = y^2 + xy^2$

$\Rightarrow x^2 - y^2 = xy^2 - x^2y$

$\Rightarrow x^2 - y^2 = xy(y-x)$

$\Rightarrow (x+y)(x-y) = xy(y-x)$

$\therefore x+y = -xy$

$\Rightarrow (1+x)y = -x$

$\Rightarrow y = \frac{-x}{(1+x)}$

Differentiating both sides with respect to *x*, we obtain

$$y = \frac{-x}{(1+x)}$$

$$\frac{dy}{dx} = -\frac{(1+x)\frac{d}{dx}(x) - x\frac{d}{dx}(1+x)}{(1+x)^2} = -\frac{(1+x)-x}{(1+x)^2} = -\frac{1}{(1+x)^2}$$

Hence, proved.

Question 15:

If $(x-a)^2 + (y-b)^2 = c^2$, for some $c > 0,$ prove that

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$ is a constant independent of *a* and *b*.

Answer

It is given that, $(x-a)^2 + (y-b)^2 = c^2$

Differentiating both sides with respect to *x*, we obtain

$$\frac{d}{dx}\left[(x-a)^2\right]+\frac{d}{dx}\left[(y-b)^2\right]=\frac{d}{dx}\left(c^2\right)$$

$$\Rightarrow 2(x-a)\cdot\frac{d}{dx}(x-a)+2(y-b)\cdot\frac{d}{dx}(y-b)=0$$

$$\Rightarrow 2(x-a)\cdot 1+2(y-b)\cdot\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{-(x-a)}{y-b} \qquad \ldots(1)$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left[\frac{-(x-a)}{y-b}\right]$$

$$= -\left[\frac{(y-b)\cdot\frac{d}{dx}(x-a)-(x-a)\cdot\frac{d}{dx}(y-b)}{(y-b)^2}\right]$$

$$= -\left[\frac{(y-b)-(x-a)\cdot\frac{dy}{dx}}{(y-b)^2}\right]$$

$$= -\left[\frac{(y-b)-(x-a)\cdot\left\{\frac{-(x-a)}{y-b}\right\}}{(y-b)^2}\right] \qquad [\text{Using } (1)]$$

$$= -\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]$$

$$\therefore \left[\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}\right] = \frac{\left[1+\frac{(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]} = \frac{\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]}$$

$$= \frac{\left[\frac{c^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\frac{c^2}{(y-b)^3}} = \frac{\frac{c^3}{(y-b)^3}}{-\frac{c^2}{(y-b)^3}}$$

$= -c$, which is constant and is independent of a and b

Hence, proved.

Question 16:

If $\cos y = x\cos(a+y)$, with $\cos a \neq \pm 1$, prove that $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$

Answer

It is given that, $\cos y = x\cos(a+y)$

$$\therefore \frac{d}{dx}[\cos y] = \frac{d}{dx}[x\cos(a+y)]$$

$$\Rightarrow -\sin y\frac{dy}{dx} = \cos(a+y)\cdot\frac{d}{dx}(x) + x\cdot\frac{d}{dx}[\cos(a+y)]$$

$$\Rightarrow -\sin y\frac{dy}{dx} = \cos(a+y) + x\cdot[-\sin(a+y)]\frac{dy}{dx}$$

$$\Rightarrow [x\sin(a+y) - \sin y]\frac{dy}{dx} = \cos(a+y) \qquad ...(1)$$

Since $\cos y = x\cos(a+y)$, $x = \frac{\cos y}{\cos(a+y)}$

Then, equation (1) reduces to

$$\left[\frac{\cos y}{\cos(a+y)}\cdot\sin(a+y) - \sin y\right]\frac{dy}{dx} = \cos(a+y)$$

$$\Rightarrow [\cos y\cdot\sin(a+y) - \sin y\cdot\cos(a+y)]\cdot\frac{dy}{dx} = \cos^2(a+y)$$

$$\Rightarrow \sin(a+y-y)\frac{dy}{dx} = \cos^2(a+b)$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos^2(a+b)}{\sin a}$$

Hence, proved.

Question 17:

If $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$, find $\frac{d^2y}{dx^2}$

Answer

It is given that, $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$

$$\therefore \frac{dx}{dt} = a \cdot \frac{d}{dt}(\cos t + t\sin t)$$

$$= a\left[-\sin t + \sin t \cdot \frac{d}{dx}(t) + t \cdot \frac{d}{dt}(\sin t)\right]$$

$$= a[-\sin t + \sin t + t\cos t] = at\cos t$$

$$\frac{dy}{dt} = a \cdot \frac{d}{dt}(\sin t - t\cos t)$$

$$= a\left[\cos t - \left\{\cos t \cdot \frac{d}{dt}(t) + t \cdot \frac{d}{dt}(\cos t)\right\}\right]$$

$$= a[\cos t - \{\cos t - t\sin t\}] = at\sin t$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{at\sin t}{at\cos t} = \tan t$$

Then, $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dx}(\tan t) = \sec^2 t \cdot \dfrac{dt}{dx}$

$$= \sec^2 t \cdot \frac{1}{at\cos t} \qquad \left[\frac{dx}{dt} = at\cos t \Rightarrow \frac{dt}{dx} = \frac{1}{at\cos t}\right]$$

$$= \frac{\sec^3 t}{at}, 0 < t < \frac{\pi}{2}$$

Question 18:

If $f(x) = |x|^3$, show that $f''(x)$ exists for all real x, and find it.

Answer

It is known that, $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

Therefore, when $x \geq 0$, $f(x) = |x|^3 = x^3$

In this case, $f'(x) = 3x^2$ and hence, $f''(x) = 6x$

When $x < 0$, $f(x) = |x|^3 = (-x)^3 = -x^3$

In this case, $f'(x) = -3x^2$ and hence, $f''(x) = -6x$

Thus, for $f(x) = |x|^3$, $f''(x)$ exists for all real x and is given by,

$$f''(x) = \begin{cases} 6x, & \text{if } x \geq 0 \\ -6x, & \text{if } x < 0 \end{cases}$$

Question 19:

Using mathematical induction prove that $\frac{d}{dx}(x^n) = nx^{n-1}$ for all positive integers n.

Answer

To prove: $P(n): \frac{d}{dx}(x^n) = nx^{n-1}$ for all positive integers n

For $n = 1$,

$$P(1): \frac{d}{dx}(x) = 1 = 1 \cdot x^{1-1}$$

$\therefore P(n)$ is true for $n = 1$

Let $P(k)$ is true for some positive integer k.

That is, $P(k): \frac{d}{dx}(x^k) = kx^{k-1}$

It has to be proved that $P(k + 1)$ is also true.

$$\begin{aligned}
\text{Consider } \frac{d}{dx}(x^{k+1}) &= \frac{d}{dx}(x \cdot x^k) \\
&= x^k \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(x^k) \qquad \text{[By applying product rule]} \\
&= x^k \cdot 1 + x \cdot k \cdot x^{k-1} \\
&= x^k + kx^k \\
&= (k+1) \cdot x^k \\
&= (k+1) \cdot x^{(k+1)-1}
\end{aligned}$$

Thus, P(k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, the statement P(n) is true for every positive integer n.

Hence, proved.

Question 20:

Using the fact that sin (A + B) = sin A cos B + cos A sin B and the differentiation, obtain the sum formula for cosines.

Answer

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

Differentiating both sides with respect to x, we obtain

$$\frac{d}{dx}\left[\sin(A+B)\right] = \frac{d}{dx}(\sin A \cos B) + \frac{d}{dx}(\cos A \sin B)$$

$$\Rightarrow \cos(A+B)\cdot\frac{d}{dx}(A+B) = \cos B\cdot\frac{d}{dx}(\sin A) + \sin A\cdot\frac{d}{dx}(\cos B) + \sin B\cdot\frac{d}{dx}(\cos A) + \cos A\cdot\frac{d}{dx}(\sin B)$$

$$\Rightarrow \cos(A+B)\cdot\frac{d}{dx}(A+B) = \cos B\cdot\cos A\frac{dA}{dx} + \sin A(-\sin B)\frac{dB}{dx} + \sin B(-\sin A)\cdot\frac{dA}{dx} + \cos A\cos B\frac{dB}{dx}$$

$$\Rightarrow \cos(A+B)\cdot\left[\frac{dA}{dx} + \frac{dB}{dx}\right] = (\cos A\cos B - \sin A\sin B)\cdot\left[\frac{dA}{dx} + \frac{dB}{dx}\right]$$

$$\therefore \cos(A+B) = \cos A\cos B - \sin A\sin B$$

Question 22:

If $y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$, prove that $\frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

Answer

$$y=\begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

$$\Rightarrow y=(mc-nb)f(x)-(lc-na)g(x)+(lb-ma)h(x)$$

Then, $\frac{dy}{dx}=\frac{d}{dx}\left[(mc-nb)f(x)\right]-\frac{d}{dx}\left[(lc-na)g(x)\right]+\frac{d}{dx}\left[(lb-ma)h(x)\right]$

$$=(mc-nb)f'(x)-(lc-na)g'(x)+(lb-ma)h'(x)$$

$$=\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

Thus, $\frac{dy}{dx}=\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

Question 23:

If $y=e^{a\cos^{-1}x}, -1\le x\le 1$, show that $(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-a^2y=0$

Answer

It is given that, $y=e^{a\cos^{-1}x}$

Taking logarithm on both the sides, we obtain

$\log y = a\cos^{-1} x \log e$

$\log y = a\cos^{-1} x$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx} = a \times \frac{-1}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-ay}{\sqrt{1-x^2}}$$

By squaring both the sides, we obtain

$$\left(\frac{dy}{dx}\right)^2 = \frac{a^2y^2}{1-x^2}$$

$$\Rightarrow \left(1-x^2\right)\left(\frac{dy}{dx}\right)^2 = a^2y^2$$

$$\left(1-x^2\right)\left(\frac{dy}{dx}\right)^2 = a^2y^2$$

Again differentiating both sides with respect to x, we obtain

$$\left(\frac{dy}{dx}\right)^2 \frac{d}{dx}\left(1-x^2\right) + \left(1-x^2\right) \times \frac{d}{dx}\left[\left(\frac{dy}{dx}\right)^2\right] = a^2\frac{d}{dx}\left(y^2\right)$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2(-2x) + \left(1-x^2\right) \times 2\frac{dy}{dx}.\frac{d^2y}{dx^2} = a^2.2y.\frac{dy}{dx}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2(-2x) + \left(1-x^2\right) \times 2\frac{dy}{dx}.\frac{d^2y}{dx^2} = a^2.2y.\frac{dy}{dx}$$

$$\Rightarrow -x\frac{dy}{dx} + \left(1-x^2\right)\frac{d^2y}{dx^2} = a^2.y \qquad \left[\frac{dy}{dx} \neq 0\right]$$

$$\Rightarrow \left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$$

Hence, proved.