Exercise 7.2

Integrate the functions in Exercises 1 to 37:

Question 1: $\frac{2x}{1+x^2}$

Answer 1:

Let $1 + x^2 = t$

$\therefore$ $2x\,dx = dt$

$$\Rightarrow \int \frac{2x}{1+x^2}dx = \int \frac{1}{t}dt$$

$$= \log|t| + C$$

$$= \log|1+x^2| + C$$

$$= \log(1+x^2) + C$$

Question 2: $\frac{(\log x)^2}{x}$

Answer 2:

Let $\log|x| = t$

$\therefore$ $\frac{1}{x}dx = dt$

$$\Rightarrow \int \frac{(\log|x|)^2}{x}dx = \int t^2 dt$$

$$= \frac{t^3}{3} + C$$

$$= \frac{(\log|x|)^3}{3} + C$$

Question 3: $\dfrac{1}{x+x\log x}$

Answer 3:

$$\frac{1}{x+x\log x}=\frac{1}{x(1+\log x)}$$

Let 1 + log x = t

$\therefore \dfrac{1}{x}dx=dt$

$$\Rightarrow \int\frac{1}{x(1+\log x)}dx=\int\frac{1}{t}dt$$
$$=\log|t|+C$$
$$=\log|1+\log x|+C$$

Question 4: sin x · sin (cos x)

Answer 4:

sin x · sin (cos x)

Let cos x = t

∴ −sin x dx = dt

$$\Rightarrow \int\sin x\cdot\sin(\cos x)\,dx=-\int\sin t\,dt$$
$$=-[-\cos t]+C$$
$$=\cos t+C$$
$$=\cos(\cos x)+C$$

Question 5: $\sin(ax+b)\cos(ax+b)$

Answer 5:

$$\sin(ax+b)\cos(ax+b)=\frac{2\sin(ax+b)\cos(ax+b)}{2}=\frac{\sin 2(ax+b)}{2}$$

Let $2(ax+b)=t$

∴ 2adx = dt

$$\Rightarrow \int\frac{\sin 2(ax+b)}{2}dx=\frac{1}{2}\int\frac{\sin t\ dt}{2a}$$

$$=\frac{1}{4a}[-\cos t]+C$$

$$=\frac{-1}{4a}\cos 2(ax+b)+C$$

Question 6: $\sqrt{ax+b}$

Answer 6:

Let ax + b = t

⇒ adx = dt

$$\therefore dx=\frac{1}{a}dt$$

$$\Rightarrow \int(ax+b)^{\frac{1}{2}}dx=\frac{1}{a}\int t^{\frac{1}{2}}dt$$

$$=\frac{1}{a}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right)+C$$

$$=\frac{2}{3a}(ax+b)^{\frac{3}{2}}+C$$

Question 7: $x\sqrt{x+2}$

Answer 7:

Let x + 2 = t

∴ dx = dt

$$\Rightarrow \int x\sqrt{x+2}\,dx = \int (t-2)\sqrt{t}\,dt$$
$$= \int \left(t^{\frac{3}{2}} - 2t^{\frac{1}{2}}\right)dt$$
$$= \int t^{\frac{3}{2}}dt - 2\int t^{\frac{1}{2}}dt$$
$$= \frac{t^{\frac{5}{2}}}{\frac{5}{2}} - 2\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$
$$= \frac{2}{5}t^{\frac{5}{2}} - \frac{4}{3}t^{\frac{3}{2}} + C$$
$$= \frac{2}{5}(x+2)^{\frac{5}{2}} - \frac{4}{3}(x+2)^{\frac{3}{2}} + C$$

Question 8: $x\sqrt{1+2x^2}$

Answer 8:

Let $1 + 2x^2 = t$

∴ 4xdx = dt

$$\Rightarrow \int x\sqrt{1+2x^2}\,dx = \int \frac{\sqrt{t}\,dt}{4}$$
$$= \frac{1}{4}\int t^{\frac{1}{2}}dt$$
$$= \frac{1}{4}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$
$$= \frac{1}{6}\left(1+2x^2\right)^{\frac{3}{2}} + C$$

Question 9: $(4x+2)\sqrt{x^2+x+1}$

Answer 9:

Let $x^2 + x + 1 = t$

$\therefore (2x + 1)dx = dt$

$$\int (4x+2)\sqrt{x^2+x+1}\,dx$$

$$= \int 2\sqrt{t}\,dt$$

$$= 2\int \sqrt{t}\,dt$$

$$= 2\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$

$$= \frac{4}{3}\left(x^2+x+1\right)^{\frac{3}{2}} + C$$

Question 10: $\frac{1}{x-\sqrt{x}}$

Answer 10:

$$\frac{1}{x-\sqrt{x}} = \frac{1}{\sqrt{x}\left(\sqrt{x}-1\right)}$$

Let $\left(\sqrt{x}-1\right) = t$

$\therefore \frac{1}{2\sqrt{x}}dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{x}\left(\sqrt{x}-1\right)}dx = \int \frac{2}{t}dt$$

$$= 2\log|t| + C$$

$$= 2\log\left|\sqrt{x}-1\right| + C$$

Question 11: $\frac{x}{\sqrt{x+4}}, x>0$

Answer 11:

Let x + 4 = t

∴ dx = dt

$$\int \frac{x}{\sqrt{x+4}}dx = \int \frac{(t-4)}{\sqrt{t}}dt$$

$$= \int \left(\sqrt{t} - \frac{4}{\sqrt{t}}\right)dt$$

$$= \frac{t^{\frac{3}{2}}}{\frac{3}{2}} - 4\left(\frac{t^{\frac{1}{2}}}{\frac{1}{2}}\right) + C$$

$$= \frac{2}{3}(t)^{\frac{3}{2}} - 8(t)^{\frac{1}{2}} + C$$

$$= \frac{2}{3}t \cdot t^{\frac{1}{2}} - 8t^{\frac{1}{2}} + C$$

$$= \frac{2}{3}t^{\frac{1}{2}}(t-12) + C$$

$$= \frac{2}{3}(x+4)^{\frac{1}{2}}(x+4-12) + C$$

$$= \frac{2}{3}\sqrt{x+4}(x-8) + C$$

Question 12: $\left(x^3-1\right)^{\frac{1}{3}}x^5$

Answer 12:

Let $x^3 - 1 = t$

$\therefore 3x^2dx = dt$

$$\Rightarrow \int\left(x^3-1\right)^{\frac{1}{3}}x^5dx = \int\left(x^3-1\right)^{\frac{1}{3}}x^3\cdot x^2dx$$

$$= \int t^{\frac{1}{3}}(t+1)\frac{dt}{3}$$

$$= \frac{1}{3}\int\left(t^{\frac{4}{3}}+t^{\frac{1}{3}}\right)dt$$

$$= \frac{1}{3}\left[\frac{t^{\frac{7}{3}}}{\frac{7}{3}}+\frac{t^{\frac{4}{3}}}{\frac{4}{3}}\right]+C$$

$$= \frac{1}{3}\left[\frac{3}{7}t^{\frac{7}{3}}+\frac{3}{4}t^{\frac{4}{3}}\right]+C$$

$$= \frac{1}{7}\left(x^3-1\right)^{\frac{7}{3}}+\frac{1}{4}\left(x^3-1\right)^{\frac{4}{3}}+C$$

Question 13: $\dfrac{x^2}{\left(2+3x^3\right)^3}$

Answer 13:

Let $2 + 3x^3 = t$

$\therefore 9x^2\,dx = dt$

$$\Rightarrow \int \frac{x^2}{\left(2+3x^3\right)^3}dx = \frac{1}{9}\int \frac{dt}{(t)^3}$$

$$= \frac{1}{9}\left[\frac{t^{-2}}{-2}\right] + C$$

$$= \frac{-1}{18}\left(\frac{1}{t^2}\right) + C$$

$$= \frac{-1}{18\left(2+3x^3\right)^2} + C$$

Question 14: $\dfrac{1}{x(\log x)^m}, \; x > 0$

Answer 14:

Let $\log x = t$

$$\frac{1}{x}dx = dt$$

$$\Rightarrow \int \frac{1}{x(\log x)^m}dx = \int \frac{dt}{(t)^m}$$

$$= \left(\frac{t^{-m+1}}{1-m}\right) + C$$

$$= \frac{(\log x)^{1-m}}{(1-m)} + C$$

Question 15: $\dfrac{x}{9-4x^2}$

Answer 15:

Let 9 – 4x² = t

∴ −8x dx = dt

$$\Rightarrow \int \frac{x}{9-4x^2}dx = \frac{-1}{8}\int \frac{1}{t}dt$$

$$= \frac{-1}{8}\log|t| + C$$

$$= \frac{-1}{8}\log\left|9-4x^2\right| + C$$

Question 16: e^{2x+3}

Answer 16:

Let 2x + 3 = t

∴ 2dx = dt

$$\Rightarrow \int e^{2x+3}dx = \frac{1}{2}\int e^t dt$$

$$= \frac{1}{2}\left(e^t\right) + C$$

$$= \frac{1}{2}e^{(2x+3)} + C$$

Question 17: $\frac{x}{e^{x^2}}$

Answer 17:

Let $x^2 = t$

$\therefore 2xdx = dt$

$$\Rightarrow \int \frac{x}{e^{x^2}}dx = \frac{1}{2}\int \frac{1}{e^t}dt$$

$$= \frac{1}{2}\int e^{-t}dt$$

$$= \frac{1}{2}\left(\frac{e^{-t}}{-1}\right) + C$$

$$= -\frac{1}{2}e^{-x^2} + C$$

$$= \frac{-1}{2e^{x^2}} + C$$

Question 18: $\frac{e^{\tan^{-1}x}}{1+x^2}$

Answer 18:

Let $\tan^{-1}x = t$

$\therefore \frac{1}{1+x^2}dx = dt$

$$\Rightarrow \int \frac{e^{\tan^{-1}x}}{1+x^2}dx = \int e^t dt$$

$$= e^t + C$$

$$= e^{\tan^{-1}x} + C$$

Question 19: $\frac{e^{2x}-1}{e^{2x}+1}$

Answer 19:

$$\frac{e^{2x}-1}{e^{2x}+1}$$

Dividing numerator and denominator by e^x, we obtain

$$\frac{\frac{(e^{2x}-1)}{e^x}}{\frac{(e^{2x}+1)}{e^x}}=\frac{e^x-e^{-x}}{e^x+e^{-x}}$$

Let $e^x+e^{-x}=t$

$$(e^x-e^{-x})dx=dt$$

$$\Rightarrow \int\frac{e^{2x}-1}{e^{2x}+1}dx=\int\frac{e^x-e^{-x}}{e^x+e^{-x}}dx$$

$$=\int\frac{dt}{t}$$

$$=\log|t|+C$$

$$=\log|e^x+e^{-x}|+C$$

Question 20: $\dfrac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}$

Answer 20:

Let $e^{2x}+e^{-2x}=t$

$\left(2e^{2x}-2e^{-2x}\right)dx=dt$

$\Rightarrow 2\left(e^{2x}-e^{-2x}\right)dx=dt$

$$\Rightarrow \int\left(\frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}\right)dx=\int\frac{dt}{2t}$$
$$=\frac{1}{2}\int\frac{1}{t}dt$$
$$=\frac{1}{2}\log|t|+C$$
$$=\frac{1}{2}\log\left|e^{2x}+e^{-2x}\right|+C$$

Question 21: $\tan^2(2x-3)$

Answer 21:

$\tan^2(2x-3)=\sec^2(2x-3)-1$

Let 2x − 3 = t

∴ 2dx = dt

$$\Rightarrow \int\tan^2(2x-3)dx=\int\left[\left(\sec^2(2x-3)\right)-1\right]dx$$
$$=\frac{1}{2}\int\left(\sec^2 t\right)dt-\int 1dx$$
$$=\frac{1}{2}\int\sec^2 t\,dt-\int 1dx$$
$$=\frac{1}{2}\tan t-x+C$$
$$=\frac{1}{2}\tan(2x-3)-x+C$$

Question 22: $\sec^2(7 - 4x)$

Answer 22:

Let $7 - 4x = t$

$\therefore -4dx = dt$

$$\therefore \int \sec^2(7-4x)\,dx = \frac{-1}{4}\int \sec^2 t\,dt$$

$$= \frac{-1}{4}(\tan t) + C$$

$$= \frac{-1}{4}\tan(7-4x) + C$$

Question 23: $\frac{\sin^{-1} x}{\sqrt{1-x^2}}$

Answer 23:

Let $\sin^{-1} x = t$

$$\frac{1}{\sqrt{1-x^2}}dx = dt$$

$$\Rightarrow \int \frac{\sin^{-1} x}{\sqrt{1-x^2}}dx = \int t\,dt$$

$$= \frac{t^2}{2} + C$$

$$= \frac{(\sin^{-1} x)^2}{2} + C$$

Question 24: $\dfrac{2\cos x-3\sin x}{6\cos x+4\sin x}$

Answer 24:

$$\frac{2\cos x-3\sin x}{6\cos x+4\sin x}=\frac{2\cos x-3\sin x}{2(3\cos x+2\sin x)}$$

Let $3\cos x+2\sin x=t$

$$(-3\sin x+2\cos x)dx=dt$$

$$\int\frac{2\cos x-3\sin x}{6\cos x+4\sin x}dx=\int\frac{dt}{2t}$$
$$=\frac{1}{2}\int\frac{1}{t}dt$$
$$=\frac{1}{2}\log|t|+C$$
$$=\frac{1}{2}\log|2\sin x+3\cos x|+C$$

Question 25: $\dfrac{1}{\cos^2 x(1-\tan x)^2}$

Answer 25:

$$\frac{1}{\cos^2 x(1-\tan x)^2}=\frac{\sec^2 x}{(1-\tan x)^2}$$

Let $(1-\tan x)=t$

$$-\sec^2 x dx=dt$$

$$\Rightarrow\int\frac{\sec^2 x}{(1-\tan x)^2}dx=\int\frac{-dt}{t^2}$$
$$=-\int t^{-2}dt$$
$$=+\frac{1}{t}+C$$
$$=\frac{1}{(1-\tan x)}+C$$

Question 26: $\frac{\cos\sqrt{x}}{\sqrt{x}}$

Answer 26:

Let $\sqrt{x} = t$

$$\frac{1}{2\sqrt{x}}dx = dt$$

$$\Rightarrow \int\frac{\cos\sqrt{x}}{\sqrt{x}}dx = 2\int\cos t\, dt$$

$$= 2\sin t + C$$

$$= 2\sin\sqrt{x} + C$$

Question 27: $\sqrt{\sin 2x}\cos 2x$

Answer 27:

Let sin 2x = t

So, 2cos 2x dx = dt

$$\Rightarrow \int\sqrt{\sin 2x}\ \cos 2x\, dx = \frac{1}{2}\int\sqrt{t}\, dt$$

$$= \frac{1}{2}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$

$$= \frac{1}{3}t^{\frac{3}{2}} + C$$

$$= \frac{1}{3}(\sin 2x)^{\frac{3}{2}} + C$$

Question 28: $\frac{\cos x}{\sqrt{1+\sin x}}$

Answer 28:

Let 1 + sin x = t

∴ cos x dx = dt

$$\Rightarrow \int \frac{\cos x}{\sqrt{1+\sin x}} dx = \int \frac{dt}{\sqrt{t}}$$

$$= \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= 2\sqrt{t} + C$$

$$= 2\sqrt{1+\sin x} + C$$

Question 29: cot x log sin x

Answer 29:

Let log sin x = t

$$\Rightarrow \frac{1}{\sin x} \cdot \cos x \, dx = dt$$

$$\therefore \cot x \, dx = dt$$

$$\Rightarrow \int \cot x \log \sin x \, dx = \int t \, dt$$

$$= \frac{t^2}{2} + C$$

$$= \frac{1}{2}(\log \sin x)^2 + C$$

Question 30: $\frac{\sin x}{1+\cos x}$

Answer 30:

Let 1 + cos x = t

∴ −sin x dx = dt

$$\Rightarrow \int \frac{\sin x}{1+\cos x} dx = \int -\frac{dt}{t}$$

$$= -\log|t| + C$$

$$= -\log|1+\cos x| + C$$

Question 31: $\frac{\sin x}{(1+\cos x)^2}$

Answer 31:

Let 1 + cos x = t

∴ −sin x dx = dt

$$\Rightarrow \int \frac{\sin x}{(1+\cos x)^2} dx = \int -\frac{dt}{t^2}$$

$$= -\int t^{-2} dt$$

$$= \frac{1}{t} + C$$

$$= \frac{1}{1+\cos x} + C$$

Question 32: $\dfrac{1}{1+\cot x}$

Answer 32:

$$\text{Let } I=\int\frac{1}{1+\cot x}dx$$
$$=\int\frac{1}{1+\frac{\cos x}{\sin x}}dx$$
$$=\int\frac{\sin x}{\sin x+\cos x}dx$$
$$=\frac{1}{2}\int\frac{2\sin x}{\sin x+\cos x}dx$$
$$=\frac{1}{2}\int\frac{(\sin x+\cos x)+(\sin x-\cos x)}{(\sin x+\cos x)}dx$$
$$=\frac{1}{2}\int 1\,dx+\frac{1}{2}\int\frac{\sin x-\cos x}{\sin x+\cos x}dx$$
$$=\frac{1}{2}(x)+\frac{1}{2}\int\frac{\sin x-\cos x}{\sin x+\cos x}dx$$

Let sin x + cos x = t ⇒ (cos x − sin x) dx = dt

$$\therefore I=\frac{x}{2}+\frac{1}{2}\int\frac{-(dt)}{t}$$
$$=\frac{x}{2}-\frac{1}{2}\log|t|+C$$
$$=\frac{x}{2}-\frac{1}{2}\log|\sin x+\cos x|+C$$

Question 33: $\frac{1}{1-\tan x}$

Answer 33:

$$\text{Let } I=\int \frac{1}{1-\tan x}dx$$
$$=\int \frac{1}{1-\frac{\sin x}{\cos x}}dx$$
$$=\int \frac{\cos x}{\cos x-\sin x}dx$$
$$=\frac{1}{2}\int \frac{2\cos x}{\cos x-\sin x}dx$$
$$=\frac{1}{2}\int \frac{(\cos x-\sin x)+(\cos x+\sin x)}{(\cos x-\sin x)}dx$$
$$=\frac{1}{2}\int 1\,dx+\frac{1}{2}\int \frac{\cos x+\sin x}{\cos x-\sin x}dx$$
$$=\frac{x}{2}+\frac{1}{2}\int \frac{\cos x+\sin x}{\cos x-\sin x}dx$$

Put cos x − sin x = t ⇒ (−sin x − cos x) dx = dt

$$\therefore I=\frac{x}{2}+\frac{1}{2}\int \frac{-(dt)}{t}$$
$$=\frac{x}{2}-\frac{1}{2}\log|t|+C$$
$$=\frac{x}{2}-\frac{1}{2}\log|\cos x-\sin x|+C$$

Question 34: $\frac{\sqrt{\tan x}}{\sin x\cos x}$

Answer 34:

$$\text{Let } I=\int\frac{\sqrt{\tan x}}{\sin x\cos x}dx$$
$$=\int\frac{\sqrt{\tan x}\times\cos x}{\sin x\cos x\times\cos x}dx$$
$$=\int\frac{\sqrt{\tan x}}{\tan x\cos^2 x}dx$$
$$=\int\frac{\sec^2 x\,dx}{\sqrt{\tan x}}$$

Let $\tan x=t \Rightarrow \sec^2 x\,dx=dt$

$$\therefore I=\int\frac{dt}{\sqrt{t}}$$
$$=2\sqrt{t}+C$$
$$=2\sqrt{\tan x}+C$$

Question 35: $\frac{(1+\log x)^2}{x}$

Answer 35:

Let 1 + log x = t

$$\therefore \frac{1}{x}dx=dt$$

$$\Rightarrow \int\frac{(1+\log x)^2}{x}dx=\int t^2 dt$$
$$=\frac{t^3}{3}+C$$
$$=\frac{(1+\log x)^3}{3}+C$$

Question 36: $\dfrac{(x+1)(x+\log x)^2}{x}$

Answer 36:

$$\frac{(x+1)(x+\log x)^2}{x}=\left(\frac{x+1}{x}\right)(x+\log x)^2=\left(1+\frac{1}{x}\right)(x+\log x)^2$$

Let $(x+\log x)=t$

$\therefore \left(1+\frac{1}{x}\right)dx=dt$

$$\Rightarrow \int\left(1+\frac{1}{x}\right)(x+\log x)^2\,dx=\int t^2\,dt$$

$$=\frac{t^3}{3}+C$$

$$=\frac{1}{3}(x+\log x)^3+C$$

Question 37: $\dfrac{x^3\sin\left(\tan^{-1}x^4\right)}{1+x^8}$

Answer 37:

Let $x^4=t$

$\therefore 4x^3\,dx=dt$

$$\Rightarrow \int\frac{x^3\sin\left(\tan^{-1}x^4\right)}{1+x^8}dx=\frac{1}{4}\int\frac{\sin\left(\tan^{-1}t\right)}{1+t^2}dt \qquad ...(1)$$

Let $\tan^{-1}t=u$

$\therefore \frac{1}{1+t^2}dt=du$

From (1), we obtain

$$\int \frac{x^3 \sin\left(\tan^{-1} x^4\right) dx}{1+x^8} = \frac{1}{4}\int \sin u \, du$$

$$= \frac{1}{4}(-\cos u) + C$$

$$= \frac{-1}{4}\cos\left(\tan^{-1} t\right) + C$$

$$= \frac{-1}{4}\cos\left(\tan^{-1} x^4\right) + C$$

Question 38:

$\int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x} dx$ equals

(A) $10^x - x^{10} + C$ (B) $10^x + x^{10} + C$

(C) $\left(10^x - x^{10}\right)^{-1} + C$ (D) $\log\left(10^x + x^{10}\right) + C$

Answer 38:

Let $x^{10} + 10^x = t$

$\therefore \left(10x^9 + 10^x \log_e 10\right) dx = dt$

$$\Rightarrow \int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x} dx = \int \frac{dt}{t}$$

$$= \log t + C$$

$$= \log\left(10^x + x^{10}\right) + C$$

Hence, the correct Answer is D.

Question 39: $\int \frac{dx}{\sin^2 x \cos^2 x}$ equals

(A) $\tan x + \cot x + C$ (B) $\tan x - \cot x + C$

(C) $\tan x \cot x + C$ (D) $\tan x - \cot 2x + C$

Answer 39:

$$\text{Let } I = \int \frac{dx}{\sin^2 x \cos^2 x}$$
$$= \int \frac{1}{\sin^2 x \cos^2 x} dx$$
$$= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx$$
$$= \int \frac{\sin^2 x}{\sin^2 x \cos^2 x} dx + \int \frac{\cos^2 x}{\sin^2 x \cos^2 x} dx$$
$$= \int \sec^2 x dx + \int \operatorname{cosec}^2 x dx$$
$$= \tan x - \cot x + C$$

Hence, the correct Answer is B.