Exercise 7.5

Integrate the rational functions in Exercises 1 to 21.

Question 1: $\frac{x}{(x+1)(x+2)}$

Answer 1:

Let $\frac{x}{(x+1)(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+2)}$

$\Rightarrow x = A(x+2) + B(x+1)$

Equating the coefficients of x and constant term, we obtain

A + B = 1

2A + B = 0

On solving, we obtain

A = −1 and B = 2

$$\therefore \frac{x}{(x+1)(x+2)} = \frac{-1}{(x+1)} + \frac{2}{(x+2)}$$

$$\Rightarrow \int \frac{x}{(x+1)(x+2)}dx = \int \frac{-1}{(x+1)} + \frac{2}{(x+2)}dx$$

$$= -\log|x+1| + 2\log|x+2| + C$$

$$= \log(x+2)^2 - \log|x+1| + C$$

$$= \log\frac{(x+2)^2}{(x+1)} + C$$

Question 2: $\frac{1}{x^2-9}$

Answer 2:

Let $\frac{1}{(x+3)(x-3)} = \frac{A}{(x+3)} + \frac{B}{(x-3)}$

$1 = A(x-3) + B(x+3)$

Equating the coefficients of x and constant term, we obtain

A + B = 0

$-3A + 3B = 1$

On solving, we obtain

$$A = -\frac{1}{6} \text{ and } B = \frac{1}{6}$$

$$\therefore \frac{1}{(x+3)(x-3)} = \frac{-1}{6(x+3)} + \frac{1}{6(x-3)}$$

$$\Rightarrow \int \frac{1}{(x^2-9)} dx = \int \left(\frac{-1}{6(x+3)} + \frac{1}{6(x-3)} \right) dx$$

$$= -\frac{1}{6} \log|x+3| + \frac{1}{6} \log|x-3| + C$$

$$= \frac{1}{6} \log \left| \frac{(x-3)}{(x+3)} \right| + C$$

Question 3: $\frac{3x-1}{(x-1)(x-2)(x-3)}$

Answer 3:

Let $\frac{3x-1}{(x-1)(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$

$$3x-1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) \quad \ldots(1)$$

Equating the coefficients of x^2, x and constant term, we obtain

$A + B + C = 0$

$-5A - 4B - 3C = 3$

$6A + 3B + 2C = -1$

Solving these equations, we obtain

$A = 1$, $B = -5$, and $C = 4$

$$\therefore \frac{3x-1}{(x-1)(x-2)(x-3)} = \frac{1}{(x-1)} - \frac{5}{(x-2)} + \frac{4}{(x-3)}$$

$$\Rightarrow \int \frac{3x-1}{(x-1)(x-2)(x-3)} dx = \int \left\{ \frac{1}{(x-1)} - \frac{5}{(x-2)} + \frac{4}{(x-3)} \right\} dx$$

$$= \log|x-1| - 5\log|x-2| + 4\log|x-3| + C$$

Question 4: $\frac{x}{(x-1)(x-2)(x-3)}$

Answer 4:

Let $\frac{x}{(x-1)(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$

$$x = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) \quad \ldots(1)$$

Equating the coefficients of x^2, x and constant term, we obtain

A + B + C = 0

− 5A − 4B − 3C = 1

6A + 4B + 2C = 0

Solving these equations, we obtain

$A = \frac{1}{2}$, $B = -2$, and $C = \frac{3}{2}$

$$\therefore \frac{x}{(x-1)(x-2)(x-3)} = \frac{1}{2(x-1)} - \frac{2}{(x-2)} + \frac{3}{2(x-3)}$$

$$\Rightarrow \int \frac{x}{(x-1)(x-2)(x-3)} dx = \int \left\{ \frac{1}{2(x-1)} - \frac{2}{(x-2)} + \frac{3}{2(x-3)} \right\} dx$$

$$= \frac{1}{2}\log|x-1| - 2\log|x-2| + \frac{3}{2}\log|x-3| + C$$

Question 5: $\dfrac{2x}{x^2+3x+2}$

Answer 5:

Let $\dfrac{2x}{x^2+3x+2}=\dfrac{A}{(x+1)}+\dfrac{B}{(x+2)}$

$2x=A(x+2)+B(x+1)$ ...(1)

Equating the coefficients of x^2, x and constant term, we obtain

A + B = 2

2A + B = 0

Solving these equations, we obtain

A = −2 and B = 4

$$\therefore \frac{2x}{(x+1)(x+2)}=\frac{-2}{(x+1)}+\frac{4}{(x+2)}$$

$$\Rightarrow \int\frac{2x}{(x+1)(x+2)}dx=\int\left\{\frac{4}{(x+2)}-\frac{2}{(x+1)}\right\}dx$$

$$=4\log|x+2|-2\log|x+1|+C$$

Question 6: $\dfrac{1-x^2}{x(1-2x)}$

Answer 6:

It can be seen that the given integrand is not a proper fraction.

Therefore, on dividing $(1-x^2)$ by $x(1-2x)$, we obtain

$$\frac{1-x^2}{x(1-2x)}=\frac{1}{2}+\frac{1}{2}\left(\frac{2-x}{x(1-2x)}\right)$$

Let $\dfrac{2-x}{x(1-2x)}=\dfrac{A}{x}+\dfrac{B}{(1-2x)}$

$\Rightarrow (2-x)=A(1-2x)+Bx$ ...(1)

Equating the coefficients of x^2, x and constant term, we obtain

– 2A + B = – 1

And A = 2

Solving these equations, we obtain

A = 2 and B = 3

$$\therefore \frac{2-x}{x(1-2x)}=\frac{2}{x}+\frac{3}{1-2x}$$

Substituting in equation (1), we obtain

$$\frac{1-x^2}{x(1-2x)}=\frac{1}{2}+\frac{1}{2}\left\{\frac{2}{x}+\frac{3}{(1-2x)}\right\}$$

$$\Rightarrow \int\frac{1-x^2}{x(1-2x)}dx=\int\left\{\frac{1}{2}+\frac{1}{2}\left(\frac{2}{x}+\frac{3}{1-2x}\right)\right\}dx$$

$$=\frac{x}{2}+\log|x|+\frac{3}{2(-2)}\log|1-2x|+C$$

$$=\frac{x}{2}+\log|x|-\frac{3}{4}\log|1-2x|+C$$

Question 7: $\frac{x}{(x^2+1)(x-1)}$

Answer 7:

Let $\frac{x}{(x^2+1)(x-1)}=\frac{Ax+B}{(x^2+1)}+\frac{C}{(x-1)}$

$$x=(Ax+B)(x-1)+C(x^2+1)$$

$$x=Ax^2-Ax+Bx-B+Cx^2+C$$

Equating the coefficients of x^2, x, and constant term, we obtain

A + C = 0

−A + B = 1

−B + C = 0

On solving these equations, we obtain

$$A=-\frac{1}{2}, B=\frac{1}{2}, \text{ and } C=\frac{1}{2}$$

From equation (1), we obtain

$$\therefore \frac{x}{(x^2+1)(x-1)} = \frac{\left(-\frac{1}{2}x+\frac{1}{2}\right)}{x^2+1} + \frac{\frac{1}{2}}{(x-1)}$$

$$\Rightarrow \int \frac{x}{(x^2+1)(x-1)} = -\frac{1}{2}\int \frac{x}{x^2+1}dx + \frac{1}{2}\int \frac{1}{x^2+1}dx + \frac{1}{2}\int \frac{1}{x-1}dx$$

$$= -\frac{1}{4}\int \frac{2x}{x^2+1}dx + \frac{1}{2}\tan^{-1}x + \frac{1}{2}\log|x-1| + C$$

Consider $\int \frac{2x}{x^2+1}dx$, let $(x^2+1) = t \Rightarrow 2x\,dx = dt$

$$\Rightarrow \int \frac{2x}{x^2+1}dx = \int \frac{dt}{t} = \log|t| = \log|x^2+1|$$

$$\therefore \int \frac{x}{(x^2+1)(x-1)} = -\frac{1}{4}\log|x^2+1| + \frac{1}{2}\tan^{-1}x + \frac{1}{2}\log|x-1| + C$$

$$= \frac{1}{2}\log|x-1| - \frac{1}{4}\log|x^2+1| + \frac{1}{2}\tan^{-1}x + C$$

Question 8: $\frac{x}{(x-1)^2(x+2)}$

Answer 8:

Let $\frac{x}{(x-1)^2(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+2)}$

$$x = A(x-1)(x+2) + B(x+2) + C(x-1)^2$$

Substituting x = 1, we obtain

Equating the coefficients of x^2, x and constant term, we obtain

A + C = 0

A + B − 2C = 1

−2A + 2B + C = 0

On solving, we obtain

$$A=\frac{2}{9} \text{ and } C=\frac{-2}{9}$$

$$B=\frac{1}{3}$$

$$\therefore \frac{x}{(x-1)^2(x+2)}=\frac{2}{9(x-1)}+\frac{1}{3(x-1)^2}-\frac{2}{9(x+2)}$$

$$\Rightarrow \int\frac{x}{(x-1)^2(x+2)}dx=\frac{2}{9}\int\frac{1}{(x-1)}dx+\frac{1}{3}\int\frac{1}{(x-1)^2}dx-\frac{2}{9}\int\frac{1}{(x+2)}dx$$

$$=\frac{2}{9}\log|x-1|+\frac{1}{3}\left(\frac{-1}{x-1}\right)-\frac{2}{9}\log|x+2|+C$$

$$=\frac{2}{9}\log\left|\frac{x-1}{x+2}\right|-\frac{1}{3(x-1)}+C$$

Question 9: $\dfrac{3x+5}{x^3-x^2-x+1}$

Answer 9:

$$\frac{3x+5}{x^3-x^2-x+1}=\frac{3x+5}{(x-1)^2(x+1)}$$

Let $\dfrac{3x+5}{(x-1)^2(x+1)}=\dfrac{A}{(x-1)}+\dfrac{B}{(x-1)^2}+\dfrac{C}{(x+1)}$

$$3x+5=A(x-1)(x+1)+B(x+1)+C(x-1)^2$$

$$3x+5=A(x^2-1)+B(x+1)+C(x^2+1-2x) \quad ...(1)$$

Equating the coefficients of x^2, x and constant term, we obtain

A + C = 0

B − 2C = 3

− A + B + C = 5

On solving, we obtain

B = 4

$$A=-\frac{1}{2} \text{ and } C=\frac{1}{2}$$

$$\therefore \frac{3x+5}{(x-1)^2(x+1)} = \frac{-1}{2(x-1)} + \frac{4}{(x-1)^2} + \frac{1}{2(x+1)}$$

$$\Rightarrow \int \frac{3x+5}{(x-1)^2(x+1)} dx = -\frac{1}{2}\int \frac{1}{x-1} dx + 4\int \frac{1}{(x-1)^2} dx + \frac{1}{2}\int \frac{1}{(x+1)} dx$$

$$= -\frac{1}{2}\log|x-1| + 4\left(\frac{-1}{x-1}\right) + \frac{1}{2}\log|x+1| + C$$

$$= \frac{1}{2}\log\left|\frac{x+1}{x-1}\right| - \frac{4}{(x-1)} + C$$

Question 10: $\frac{2x-3}{(x^2-1)(2x+3)}$

Answer 10:

$$\frac{2x-3}{(x^2-1)(2x+3)} = \frac{2x-3}{(x+1)(x-1)(2x+3)}$$

Let $\frac{2x-3}{(x+1)(x-1)(2x+3)} = \frac{A}{(x+1)} + \frac{B}{(x-1)} + \frac{C}{(2x+3)}$

$$\Rightarrow (2x-3) = A(x-1)(2x+3) + B(x+1)(2x+3) + C(x+1)(x-1)$$

$$\Rightarrow (2x-3) = A(2x^2+x-3) + B(2x^2+5x+3) + C(x^2-1)$$

$$\Rightarrow (2x-3) = (2A+2B+C)x^2 + (A+5B)x + (-3A+3B-C)$$

Equating the coefficients of x^2, x and constant, we obtain

2A + 2B + C = 0
A + 5B = 2
− 3A + 3B − C = − 3

On solving, we obtain

$$B = -\frac{1}{10},\ A = \frac{5}{2},\ \text{and } C = -\frac{24}{5}$$

$$\therefore \frac{2x-3}{(x+1)(x-1)(2x+3)} = \frac{5}{2(x+1)} - \frac{1}{10(x-1)} - \frac{24}{5(2x+3)}$$

$$\Rightarrow \int \frac{2x-3}{(x^2-1)(2x+3)}dx = \frac{5}{2}\int \frac{1}{(x+1)}dx - \frac{1}{10}\int \frac{1}{x-1}dx - \frac{24}{5}\int \frac{1}{(2x+3)}dx$$

$$= \frac{5}{2}\log|x+1| - \frac{1}{10}\log|x-1| - \frac{24}{5\times 2}\log|2x+3|$$

$$= \frac{5}{2}\log|x+1| - \frac{1}{10}\log|x-1| - \frac{12}{5}\log|2x+3| + C$$

Question 11: $\frac{5x}{(x+1)(x^2-4)}$

Answer 11:

$$\frac{5x}{(x+1)(x^2-4)} = \frac{5x}{(x+1)(x+2)(x-2)}$$

Let $\frac{5x}{(x+1)(x+2)(x-2)} = \frac{A}{(x+1)} + \frac{B}{(x+2)} + \frac{C}{(x-2)}$

$$5x = A(x+2)(x-2) + B(x+1)(x-2) + C(x+1)(x+2) \quad ...(1)$$

Equating the coefficients of x^2, x and constant, we obtain

A + B + C = 0

− B + 3C = 5 and

− 4A − 2B + 2C = 0

On solving, we obtain

$A = \frac{5}{3}, B = -\frac{5}{2}, \text{ and } C = \frac{5}{6}$

$$\therefore \frac{5x}{(x+1)(x+2)(x-2)} = \frac{5}{3(x+1)} - \frac{5}{2(x+2)} + \frac{5}{6(x-2)}$$

$$\Rightarrow \int \frac{5x}{(x+1)(x^2-4)}dx = \frac{5}{3}\int \frac{1}{(x+1)}dx - \frac{5}{2}\int \frac{1}{(x+2)}dx + \frac{5}{6}\int \frac{1}{(x-2)}dx$$

$$= \frac{5}{3}\log|x+1| - \frac{5}{2}\log|x+2| + \frac{5}{6}\log|x-2| + C$$

Question 12: $\dfrac{x^3+x+1}{x^2-1}$

Answer 12:

It can be seen that the given integrand is not a proper fraction.

Therefore, on dividing ($x^3 + x + 1$) by $x^2 - 1$, we obtain

$$\frac{x^3+x+1}{x^2-1}=x+\frac{2x+1}{x^2-1}$$

Let $\dfrac{2x+1}{x^2-1}=\dfrac{A}{(x+1)}+\dfrac{B}{(x-1)}$

$$2x+1=A(x-1)+B(x+1) \qquad ...(1)$$

Equating the coefficients of x and constant, we obtain

A + B = 2

− A + B = 1

On solving, we obtain

$$A=\frac{1}{2} \text{ and } B=\frac{3}{2}$$

$$\therefore \frac{x^3+x+1}{x^2-1}=x+\frac{1}{2(x+1)}+\frac{3}{2(x-1)}$$

$$\Rightarrow \int\frac{x^3+x+1}{x^2-1}dx=\int x\,dx+\frac{1}{2}\int\frac{1}{(x+1)}dx+\frac{3}{2}\int\frac{1}{(x-1)}dx$$

$$=\frac{x^2}{2}+\frac{1}{2}\log|x+1|+\frac{3}{2}\log|x-1|+C$$

Question 13: $\dfrac{2}{(1-x)(1+x^2)}$

Answer 13:

$$\text{Let } \frac{2}{(1-x)(1+x^2)} = \frac{A}{(1-x)} + \frac{Bx+C}{(1+x^2)}$$

$$2 = A(1+x^2) + (Bx+C)(1-x)$$

$$2 = A + Ax^2 + Bx - Bx^2 + C - Cx$$

Equating the coefficient of x^2, x, and constant term, we obtain

A − B = 0

B − C = 0

A + C = 2

On solving these equations, we obtain

A = 1, B = 1, and C = 1

$$\therefore \frac{2}{(1-x)(1+x^2)} = \frac{1}{1-x} + \frac{x+1}{1+x^2}$$

$$\Rightarrow \int \frac{2}{(1-x)(1+x^2)}dx = \int \frac{1}{1-x}dx + \int \frac{x}{1+x^2}dx + \int \frac{1}{1+x^2}dx$$

$$= -\int \frac{1}{x-1}dx + \frac{1}{2}\int \frac{2x}{1+x^2}dx + \int \frac{1}{1+x^2}dx$$

$$= -\log|x-1| + \frac{1}{2}\log|1+x^2| + \tan^{-1}x + C$$

Question 14: $\frac{3x-1}{(x+2)^2}$

Answer 14:

$$\text{Let } \frac{3x-1}{(x+2)^2} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2}$$

$$\Rightarrow 3x-1 = A(x+2) + B$$

Equating the coefficient of x and constant term, we obtain

A = 3

2A + B = −1 ⇒ B = −7

$$\therefore \frac{3x-1}{(x+2)^2} = \frac{3}{(x+2)} - \frac{7}{(x+2)^2}$$

$$\Rightarrow \int \frac{3x-1}{(x+2)^2}dx = 3\int \frac{1}{(x+2)}dx - 7\int \frac{x}{(x+2)^2}dx$$

$$= 3\log|x+2| - 7\left(\frac{-1}{(x+2)}\right) + C$$

$$= 3\log|x+2| + \frac{7}{(x+2)} + C$$

Question 15: $\frac{1}{x^4-1}$

Answer 15:

$$\frac{1}{(x^4-1)} = \frac{1}{(x^2-1)(x^2+1)} = \frac{1}{(x+1)(x-1)(1+x^2)}$$

$$\text{Let } \frac{1}{(x+1)(x-1)(1+x^2)} = \frac{A}{(x+1)} + \frac{B}{(x-1)} + \frac{Cx+D}{(x^2+1)}$$

$$1 = A(x-1)(x^2+1) + B(x+1)(x^2+1) + (Cx+D)(x^2-1)$$

$$1 = A(x^3+x-x^2-1) + B(x^3+x+x^2+1) + Cx^3 + Dx^2 - Cx - D$$

$$1 = (A+B+C)x^3 + (-A+B+D)x^2 + (A+B-C)x + (-A+B-D)$$

Equating the coefficient of x^3, x^2, x, and constant term, we obtain

$A+B+C=0$

$-A+B+D=0$

$A+B-C=0$

$-A+B-D=1$

On solving these equations, we obtain

$A=-\frac{1}{4}, B=\frac{1}{4}, C=0, \text{ and } D=-\frac{1}{2}$

$\therefore \frac{1}{x^4-1}=\frac{-1}{4(x+1)}+\frac{1}{4(x-1)}-\frac{1}{2(x^2+1)}$

$\Rightarrow \int\frac{1}{x^4-1}dx=-\frac{1}{4}\log|x-1|+\frac{1}{4}\log|x-1|-\frac{1}{2}\tan^{-1}x+C$

$=\frac{1}{4}\log\left|\frac{x-1}{x+1}\right|-\frac{1}{2}\tan^{-1}x+C$

Question 16: $\frac{1}{x(x^n+1)}$

[Hint: multiply numerator and denominator by x^{n-1} and put $x^n = t$]

Answer 16:

$\frac{1}{x(x^n+1)}$

Multiplying numerator and denominator by x^{n-1}, we obtain

$\frac{1}{x(x^n+1)}=\frac{x^{n-1}}{x^{n-1}x(x^n+1)}=\frac{x^{n-1}}{x^n(x^n+1)}$

$\text{Let } x^n=t \Rightarrow x^{n-1}dx=dt$

$\therefore \int\frac{1}{x(x^n+1)}dx=\int\frac{x^{n-1}}{x^n(x^n+1)}dx=\frac{1}{n}\int\frac{1}{t(t+1)}dt$

$\text{Let } \frac{1}{t(t+1)}=\frac{A}{t}+\frac{B}{(t+1)}$

$1=A(1+t)+Bt \qquad ...(1)$

Equating the coefficients of t and constant, we obtain

A = 1 and B = −1

$$\therefore \frac{1}{t(t+1)}=\frac{1}{t}-\frac{1}{(1+t)}$$

$$\Rightarrow \int \frac{1}{x\left(x^n+1\right)}dx=\frac{1}{n}\int\left\{\frac{1}{t}-\frac{1}{(t+1)}\right\}dx$$

$$=\frac{1}{n}\left[\log|t|-\log|t+1|\right]+C$$

$$=-\frac{1}{n}\left[\log\left|x^n\right|-\log\left|x^n+1\right|\right]+C$$

$$=\frac{1}{n}\log\left|\frac{x^n}{x^n+1}\right|+C$$

Question 17: $\dfrac{\cos x}{(1-\sin x)(2-\sin x)}$ [Hint: Put sin x = t]

Answer 17:

$$\frac{\cos x}{(1-\sin x)(2-\sin x)}$$

Let $\sin x=t \Rightarrow \cos x\,dx=dt$

$$\therefore \int\frac{\cos x}{(1-\sin x)(2-\sin x)}dx=\int\frac{dt}{(1-t)(2-t)}$$

Let $\dfrac{1}{(1-t)(2-t)}=\dfrac{A}{(1-t)}+\dfrac{B}{(2-t)}$

$$1=A(2-t)+B(1-t) \qquad \text{...(1)}$$

Equating the coefficients of **t** and constant, we obtain

− 2A − B = 0 and

2A + B = 1

On solving, we obtain

A = 1 and B = −1

$$\therefore \frac{1}{(1-t)(2-t)}=\frac{1}{(1-t)}-\frac{1}{(2-t)}$$

$$\Rightarrow \int \frac{\cos x}{(1-\sin x)(2-\sin x)}dx = \int \left\{ \frac{1}{1-t} - \frac{1}{(2-t)} \right\} dt$$

$$= -\log|1-t| + \log|2-t| + C$$

$$= \log\left|\frac{2-t}{1-t}\right| + C$$

$$= \log\left|\frac{2-\sin x}{1-\sin x}\right| + C$$

Question 18: $\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$

Answer 18:

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = 1 - \frac{(4x^2+10)}{(x^2+3)(x^2+4)}$$

$$\text{Let } \frac{4x^2+10}{(x^2+3)(x^2+4)} = \frac{Ax+B}{(x^2+3)} + \frac{Cx+D}{(x^2+4)}$$

$$4x^2+10 = (Ax+B)(x^2+4) + (Cx+D)(x^2+3)$$

$$4x^2+10 = Ax^3 + 4Ax + Bx^2 + 4B + Cx^3 + 3Cx + Dx^2 + 3D$$

$$4x^2+10 = (A+C)x^3 + (B+D)x^2 + (4A+3C)x + (4B+3D)$$

Equating the coefficients of x^3, x^2, x, and constant term, we obtain

A + C = 0

B + D = 4

4A + 3C = 0

4B + 3D = 10

On solving these equations, we obtain

A = 0, B = −2, C = 0, and D = 6

$$\therefore \frac{4x^2+10}{(x^2+3)(x^2+4)} = \frac{-2}{(x^2+3)} + \frac{6}{(x^2+4)}$$

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}=1-\left(\frac{-2}{(x^2+3)}+\frac{6}{(x^2+4)}\right)$$

$$\Rightarrow \int\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}dx=\int\left\{1+\frac{2}{(x^2+3)}-\frac{6}{(x^2+4)}\right\}dx$$

$$=\int\left\{1+\frac{2}{x^2+(\sqrt{3})^2}-\frac{6}{x^2+2^2}\right\}$$

$$=x+2\left(\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}\right)-6\left(\frac{1}{2}\tan^{-1}\frac{x}{2}\right)+C$$

$$=x+\frac{2}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}-3\tan^{-1}\frac{x}{2}+C$$

Question 19: $\frac{2x}{(x^2+1)(x^2+3)}$

Answer 19:

$$\frac{2x}{(x^2+1)(x^2+3)}$$

Let $x^2 = t \Rightarrow 2x\,dx = dt$

$$\therefore \int\frac{2x}{(x^2+1)(x^2+3)}dx=\int\frac{dt}{(t+1)(t+3)} \quad ...(1)$$

$$\text{Let } \frac{1}{(t+1)(t+3)}=\frac{A}{(t+1)}+\frac{B}{(t+3)}$$

$$1=A(t+3)+B(t+1) \quad ...(1)$$

Equating the coefficients of *t* and constant, we obtain

A + B = 0 and 3A + B = 1

On solving, we obtain

$A=\frac{1}{2}$ and $B=-\frac{1}{2}$

$$\therefore \frac{1}{(t+1)(t+3)}=\frac{1}{2(t+1)}-\frac{1}{2(t+3)}$$

$$\Rightarrow \int\frac{2x}{(x^2+1)(x^2+3)}dx=\int\left\{\frac{1}{2(t+1)}-\frac{1}{2(t+3)}\right\}dt$$

$$=\frac{1}{2}\log|(t+1)|-\frac{1}{2}\log|t+3|+C$$

$$=\frac{1}{2}\log\left|\frac{t+1}{t+3}\right|+C$$

$$=\frac{1}{2}\log\left|\frac{x^2+1}{x^2+3}\right|+C$$

Question 20: $\frac{1}{x(x^4-1)}$

Answer 20:

$$\frac{1}{x(x^4-1)}$$

Multiplying numerator and denominator by x^3, we obtain

$$\frac{1}{x(x^4-1)}=\frac{x^3}{x^4(x^4-1)}$$

$$\therefore \int\frac{1}{x(x^4-1)}dx=\int\frac{x^3}{x^4(x^4-1)}dx$$

Let $x^4 = t \Rightarrow 4x^3dx = dt$

$$\therefore \int\frac{1}{x(x^4-1)}dx=\frac{1}{4}\int\frac{dt}{t(t-1)}$$

$$\text{Let } \frac{1}{t(t-1)}=\frac{A}{t}+\frac{B}{(t-1)}$$

$$1=A(t-1)+Bt \qquad ...(1)$$

Equating the coefficients of t and constant, we obtain

A = −1 and B = 1

$$\Rightarrow \frac{1}{t(t+1)} = \frac{-1}{t} + \frac{1}{t-1}$$

$$\Rightarrow \int \frac{1}{x(x^4-1)} dx = \frac{1}{4} \int \left\{ \frac{-1}{t} + \frac{1}{t-1} \right\} dt$$

$$= \frac{1}{4}\left[-\log|t| + \log|t-1|\right] + C$$

$$= \frac{1}{4} \log\left|\frac{t-1}{t}\right| + C$$

$$= \frac{1}{4} \log\left|\frac{x^4-1}{x^4}\right| + C$$

Question 21: $\frac{1}{(e^x-1)}$ [Hint: Put $e^x = t$]

Answer 21:

$$\frac{1}{(e^x-1)}$$

Let $e^x = t \Rightarrow e^x\, dx = dt$

$$\Rightarrow \int \frac{1}{e^x-1} dx = \int \frac{1}{t-1} \times \frac{dt}{t} = \int \frac{1}{t(t-1)} dt$$

$$\text{Let } \frac{1}{t(t-1)} = \frac{A}{t} + \frac{B}{t-1}$$

$$1 = A(t-1) + Bt \qquad \ldots(1)$$

Equating the coefficients of *t* and constant, we obtain

A = −1 and B = 1

$$\therefore \frac{1}{t(t-1)} = \frac{-1}{t} + \frac{1}{t-1}$$

$$\Rightarrow \int \frac{1}{t(t-1)} dt = \log\left|\frac{t-1}{t}\right| + C$$

$$= \log\left|\frac{e^x - 1}{e^x}\right| + C$$

Question 22:

$\int \frac{x dx}{(x-1)(x-2)}$ equals

A. $\log\left|\frac{(x-1)^2}{x-2}\right| + C$

B. $\log\left|\frac{(x-2)^2}{x-1}\right| + C$

C. $\log\left|\left(\frac{x-1}{x-2}\right)^2\right| + C$

D. $\log|(x-1)(x-2)| + C$

Answer 22:

Let $\frac{x}{(x-1)(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-2)}$

$x = A(x-2) + B(x-1)$ ...(1)

Equating the coefficients of x and constant, we obtain
A = −1 and B = 2

$$\therefore \frac{x}{(x-1)(x-2)} = -\frac{1}{(x-1)} + \frac{2}{(x-2)}$$

$$\Rightarrow \int \frac{x}{(x-1)(x-2)} dx = \int \left\{ \frac{-1}{(x-1)} + \frac{2}{(x-2)} \right\} dx$$

$$= -\log|x-1| + 2\log|x-2| + C$$

$$= \log\left|\frac{(x-2)^2}{x-1}\right| + C$$

Hence, the correct Answer is B.

Question 23:

$\int \frac{dx}{x(x^2+1)}$ equals

A. $\log|x| - \frac{1}{2}\log(x^2+1) + C$

B. $\log|x| + \frac{1}{2}\log(x^2+1) + C$

C. $-\log|x| + \frac{1}{2}\log(x^2+1) + C$

D. $\frac{1}{2}\log|x| + \log(x^2+1) + C$

Answer 23:

Let $\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$

$1 = A(x^2+1) + (Bx+C)x$

Equating the coefficients of x^2, x, and constant term, we obtain

A + B = 0

C = 0

A = 1

On solving these equations, we obtain

A = 1, B = −1, and C = 0

$$\therefore \frac{1}{x\left(x^2+1\right)}=\frac{1}{x}+\frac{-x}{x^2+1}$$

$$\Rightarrow \int \frac{1}{x\left(x^2+1\right)}dx=\int\left\{\frac{1}{x}-\frac{x}{x^2+1}\right\}dx$$

$$=\log|x|-\frac{1}{2}\log\left|x^2+1\right|+C$$

Hence, the correct Answer is A.