Exercise 9.3

Question 1:

$$\frac{dy}{dx}=\frac{1-\cos x}{1+\cos x}$$

Answer

The given differential equation is:

$$\frac{dy}{dx}=\frac{1-\cos x}{1+\cos x}$$

$$\Rightarrow \frac{dy}{dx}=\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}=\tan^2\frac{x}{2}$$

$$\Rightarrow \frac{dy}{dx}=\left(\sec^2\frac{x}{2}-1\right)$$

Separating the variables,we get:

$$dy=\left(\sec^2\frac{x}{2}-1\right)dx$$

Now, integrating both sides of this equation, we get:

$$\int dy=\int\left(\sec^2\frac{x}{2}-1\right)dx=\int\sec^2\frac{x}{2}dx-\int dx$$

$$\Rightarrow y=2\tan\frac{x}{2}-x+C$$

This is the required general solution of the given differential equation.

Question 2:

$$\frac{dy}{dx}=\sqrt{4-y^2}\ (-2<y<2)$$

Answer

The given differential equation is:

$$\frac{dy}{dx}=\sqrt{4-y^2}$$

Separating the variables, we get:

$$\Rightarrow \frac{dy}{\sqrt{4-y^2}}=dx$$

Now, integrating both sides of this equation, we get:

$$\int \frac{dy}{\sqrt{4-y^2}}=\int dx$$

$$\Rightarrow \sin^{-1}\frac{y}{2}=x+C$$

$$\Rightarrow \frac{y}{2}=\sin(x+C)$$

$$\Rightarrow y=2\sin(x+C)$$

This is the required general solution of the given differential equation.

Question 3:

$$\frac{dy}{dx}+y=1\ (y\neq 1)$$

Answer

The given differential equation is:

$$\frac{dy}{dx}+y=1$$

$$\Rightarrow dy+y\ dx=dx$$

$$\Rightarrow dy=(1-y)dx$$

Separating the variables, we get:

$$\Rightarrow \frac{dy}{1-y}=dx$$

Now, integrating both sides, we get:

$$\int \frac{dy}{1-y} = \int dx$$

$$\Rightarrow \log(1-y) = x + \log C$$

$$\Rightarrow -\log C - \log(1-y) = x$$

$$\Rightarrow \log C(1-y) = -x$$

$$\Rightarrow C(1-y) = e^{-x}$$

$$\Rightarrow 1-y = \frac{1}{C}e^{-x}$$

$$\Rightarrow y = 1 - \frac{1}{C}e^{-x}$$

$$\Rightarrow y = 1 + Ae^{-x} \text{ (where } A = -\frac{1}{C})$$

This is the required general solution of the given differential equation.

Question 4:

$$\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$$

Answer

The given differential equation is:

$$\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$$

$$\Rightarrow \frac{\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy}{\tan x \tan y} = 0$$

$$\Rightarrow \frac{\sec^2 x}{\tan x} dx + \frac{\sec^2 y}{\tan y} dy = 0$$

$$\Rightarrow \frac{\sec^2 x}{\tan x} dx = -\frac{\sec^2 y}{\tan y} dy$$

Integrating both sides of this equation, we get:

$$\int \frac{\sec^2 x}{\tan x} dx = -\int \frac{\sec^2 y}{\tan y} dy \quad \ldots(1)$$

Let $\tan x = t$.

$$\therefore \frac{d}{dx}(\tan x) = \frac{dt}{dx}$$

$$\Rightarrow \sec^2 x = \frac{dt}{dx}$$

$$\Rightarrow \sec^2 x\,dx = dt$$

Now, $\int \frac{\sec^2 x}{\tan x}dx = \int \frac{1}{t}dt.$

$$= \log t$$

$$= \log(\tan x)$$

Similarly, $\int \frac{\sec^2 x}{\tan x}dy = \log(\tan y).$

Substituting these values in equation (1), we get:

$$\log(\tan x) = -\log(\tan y) + \log C$$

$$\Rightarrow \log(\tan x) = \log\left(\frac{C}{\tan y}\right)$$

$$\Rightarrow \tan x = \frac{C}{\tan y}$$

$$\Rightarrow \tan x \tan y = C$$

This is the required general solution of the given differential equation.

Question 5:

$$\left(e^x + e^{-x}\right)dy - \left(e^x - e^{-x}\right)dx = 0$$

Answer

The given differential equation is:

$$\left(e^x + e^{-x}\right)dy - \left(e^x - e^{-x}\right)dx = 0$$

$$\Rightarrow \left(e^x + e^{-x}\right)dy = \left(e^x - e^{-x}\right)dx$$

$$\Rightarrow dy = \left[\frac{e^x - e^{-x}}{e^x + e^{-x}}\right]dx$$

Integrating both sides of this equation, we get:

$$\int dy = \int\left[\frac{e^x - e^{-x}}{e^x + e^{-x}}\right]dx + C$$

$$\Rightarrow y = \int\left[\frac{e^x - e^{-x}}{e^x + e^{-x}}\right]dx + C \qquad ...(1)$$

Let $(e^x + e^{-x}) = t$.

Differentiating both sides with respect to x, we get:

$$\frac{d}{dx}\left(e^x + e^{-x}\right) = \frac{dt}{dx}$$

$$\Rightarrow e^x - e^{-x} = \frac{dt}{dt}$$

$$\Rightarrow \left(e^x - e^{-x}\right)dx = dt$$

Substituting this value in equation (1), we get:

$$y = \int \frac{1}{t}dt + C$$

$$\Rightarrow y = \log(t) + C$$

$$\Rightarrow y = \log\left(e^x + e^{-x}\right) + C$$

This is the required general solution of the given differential equation.

Question 6:

$$\frac{dy}{dx} = \left(1 + x^2\right)\left(1 + y^2\right)$$

Answer

The given differential equation is:

$$\frac{dy}{dx} = \left(1 + x^2\right)\left(1 + y^2\right)$$

$$\Rightarrow \frac{dy}{1 + y^2} = \left(1 + x^2\right)dx$$

Integrating both sides of this equation, we get:

$$\int \frac{dy}{1+y^2} = \int (1+x^2)dx$$
$$\Rightarrow \tan^{-1} y = \int dx + \int x^2 dx$$
$$\Rightarrow \tan^{-1} y = x + \frac{x^3}{3} + C$$

This is the required general solution of the given differential equation.

Question 7:

$$y \log y \, dx - x \, dy = 0$$

Answer

The given differential equation is:

$$y \log y \, dx - x \, dy = 0$$
$$\Rightarrow y \log y \, dx = x \, dy$$
$$\Rightarrow \frac{dy}{y \log y} = \frac{dx}{x}$$

Integrating both sides, we get:

$$\int \frac{dy}{y \log y} = \int \frac{dx}{x} \quad ...(1)$$

Let $\log y = t$.

$$\therefore \frac{d}{dy}(\log y) = \frac{dt}{dy}$$
$$\Rightarrow \frac{1}{y} = \frac{dt}{dy}$$
$$\Rightarrow \frac{1}{y} dy = dt$$

Substituting this value in equation (1), we get:

$$\int \frac{dt}{t} = \int \frac{dx}{x}$$
$$\Rightarrow \log t = \log x + \log C$$
$$\Rightarrow \log(\log y) = \log Cx$$
$$\Rightarrow \log y = Cx$$
$$\Rightarrow y = e^{Cx}$$

This is the required general solution of the given differential equation.

Question 8:

$$x^5 \frac{dy}{dx} = -y^5$$

Answer

The given differential equation is:

$$x^5 \frac{dy}{dx} = -y^5$$
$$\Rightarrow \frac{dy}{y^5} = -\frac{dx}{x^5}$$
$$\Rightarrow \frac{dx}{x^5} + \frac{dy}{y^5} = 0$$

Integrating both sides, we get:

$$\int \frac{dx}{x^5} + \int \frac{dy}{y^5} = k \quad \text{(where } k \text{ is any constant)}$$
$$\Rightarrow \int x^{-5} dx + \int y^{-5} dy = k$$
$$\Rightarrow \frac{x^{-4}}{-4} + \frac{y^{-4}}{-4} = k$$
$$\Rightarrow x^{-4} + y^{-4} = -4k$$
$$\Rightarrow x^{-4} + y^{-4} = C \qquad (C = -4k)$$

This is the required general solution of the given differential equation.

Question 9:

$$\frac{dy}{dx}=\sin^{-1}x$$

Answer

The given differential equation is:

$$\frac{dy}{dx}=\sin^{-1}x$$

$$\Rightarrow dy=\sin^{-1}x\ dx$$

Integrating both sides, we get:

$$\int dy=\int\sin^{-1}x\,dx$$

$$\Rightarrow y=\int\left(\sin^{-1}x\cdot 1\right)dx$$

$$\Rightarrow y=\sin^{-1}x\cdot\int(1)\,dx-\int\left[\left(\frac{d}{dx}\left(\sin^{-1}x\right)\cdot\int(1)\,dx\right)\right]dx$$

$$\Rightarrow y=\sin^{-1}x\cdot x-\int\left(\frac{1}{\sqrt{1-x^2}}\cdot x\right)dx$$

$$\Rightarrow y=x\sin^{-1}x+\int\frac{-x}{\sqrt{1-x^2}}dx \quad ...(1)$$

Let $1-x^2=t$.

$$\Rightarrow\frac{d}{dx}\left(1-x^2\right)=\frac{dt}{dx}$$

$$\Rightarrow -2x=\frac{dt}{dx}$$

$$\Rightarrow x\,dx=-\frac{1}{2}dt$$

Substituting this value in equation (1), we get:

$$y = x\sin^{-1}x + \int \frac{1}{2\sqrt{t}}dt$$

$$\Rightarrow y = x\sin^{-1}x + \frac{1}{2}\cdot\int (t)^{-\frac{1}{2}}dt$$

$$\Rightarrow y = x\sin^{-1}x + \frac{1}{2}\cdot\frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$\Rightarrow y = x\sin^{-1}x + \sqrt{t} + C$$

$$\Rightarrow y = x\sin^{-1}x + \sqrt{1-x^2} + C$$

This is the required general solution of the given differential equation.

Question 10:

$$e^x \tan y\,dx + (1-e^x)\sec^2 y\,dy = 0$$

Answer

The given differential equation is:

$$e^x \tan y\,dx + (1-e^x)\sec^2 y\,dy = 0$$

$$(1-e^x)\sec^2 y\,dy = -e^x \tan y\,dx$$

Separating the variables, we get:

$$\frac{\sec^2 y}{\tan y}dy = \frac{-e^x}{1-e^x}dx$$

Integrating both sides, we get:

$$\int\frac{\sec^2 y}{\tan y}dy = \int\frac{-e^x}{1-e^x}dx \qquad ...(1)$$

Let $\tan y = u$.

$$\Rightarrow \frac{d}{dy}(\tan y) = \frac{du}{dy}$$

$$\Rightarrow \sec^2 y = \frac{du}{dy}$$

$$\Rightarrow \sec^2 y dy = du$$

$$\therefore \int\frac{\sec^2 y}{\tan y}dy = \int\frac{du}{u} = \log u = \log(\tan y)$$

Now, let $1-e^x=t$.

$\therefore \frac{d}{dx}\left(1-e^x\right)=\frac{dt}{dx}$

$\Rightarrow -e^x=\frac{dt}{dx}$

$\Rightarrow -e^x dx=dt$

$\Rightarrow \int\frac{-e^x}{1-e^x}dx=\int\frac{dt}{t}=\log t=\log\left(1-e^x\right)$

Substituting the values of $\int\frac{\sec^2 y}{\tan y}dy$ and $\int\frac{-e^x}{1-e^x}dx$

$\Rightarrow \log\left(\tan y\right)=\log\left(1-e^x\right)+\log C$

$\Rightarrow \log\left(\tan y\right)=\log\left[C\left(1-e^x\right)\right]$

$\Rightarrow \tan y=C\left(1-e^x\right)$

This is the required general solution of the given differential equation.

Question 11:

$\left(x^3+x^2+x+1\right)\frac{dy}{dx}=2x^2+x; y=1$ when $x=0$

Answer

The given differential equation is:

$\left(x^3+x^2+x+1\right)\frac{dy}{dx}=2x^2+x$

$\Rightarrow \frac{dy}{dx}=\frac{2x^2+x}{\left(x^3+x^2+x+1\right)}$

$\Rightarrow dy=\frac{2x^2+x}{(x+1)\left(x^2+1\right)}dx$

Integrating both sides, we get:

$$\int dy = \int \frac{2x^2 + x}{(x+1)(x^2+1)} dx \quad ...(1)$$

$$\text{Let } \frac{2x^2 + x}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}. \quad ...(2)$$

$$\Rightarrow \frac{2x^2 + x}{(x+1)(x^2+1)} = \frac{Ax^2 + A + (Bx+C)(x+1)}{(x+1)(x^2+1)}$$

$$\Rightarrow 2x^2 + x = Ax^2 + A + Bx^2 + Bx + Cx + C$$

$$\Rightarrow 2x^2 + x = (A+B)x^2 + (B+C)x + (A+C)$$

Comparing the coefficients of x^2 and x, we get:

A + B = 2

B + C = 1

A + C = 0

Solving these equations, we get:

$$A = \frac{1}{2},\ B = \frac{3}{2} \text{ and } C = \frac{-1}{2}$$

Substituting the values of A, B, and C in equation (2), we get:

$$\frac{2x^2 + x}{(x+1)(x^2+1)} = \frac{1}{2} \cdot \frac{1}{(x+1)} + \frac{1}{2} \frac{(3x-1)}{(x^2+1)}$$

Therefore, equation (1) becomes:

$$\int dy = \frac{1}{2}\int \frac{1}{x+1}dx + \frac{1}{2}\int \frac{3x-1}{x^2+1}dx$$

$$\Rightarrow y = \frac{1}{2}\log(x+1) + \frac{3}{2}\int \frac{x}{x^2+1}dx - \frac{1}{2}\int \frac{1}{x^2+1}dx$$

$$\Rightarrow y = \frac{1}{2}\log(x+1) + \frac{3}{4}\cdot\int \frac{2x}{x^2+1}dx - \frac{1}{2}\tan^{-1}x + C$$

$$\Rightarrow y = \frac{1}{2}\log(x+1) + \frac{3}{4}\log(x^2+1) - \frac{1}{2}\tan^{-1}x + C$$

$$\Rightarrow y = \frac{1}{4}\left[2\log(x+1) + 3\log(x^2+1)\right] - \frac{1}{2}\tan^{-1}x + C$$

$$\Rightarrow y = \frac{1}{4}\left[(x+1)^2(x^2+1)^3\right] - \frac{1}{2}\tan^{-1}x + C \quad ...(3)$$

Now, $y = 1$ when $x = 0$.

$$\Rightarrow 1 = \frac{1}{4}\log(1) - \frac{1}{2}\tan^{-1}0 + C$$

$$\Rightarrow 1 = \frac{1}{4}\times 0 - \frac{1}{2}\times 0 + C$$

$$\Rightarrow C = 1$$

Substituting C = 1 in equation (3), we get:

$$y = \frac{1}{4}\left[\log(x+1)^2(x^2+1)^3\right] - \frac{1}{2}\tan^{-1}x + 1$$

Question 12:

$$x(x^2-1)\frac{dy}{dx} = 1;\ y = 0 \text{ when } x = 2$$

Answer

$$x(x^2-1)\frac{dy}{dx} = 1$$

$$\Rightarrow dy = \frac{dx}{x(x^2-1)}$$

$$\Rightarrow dy = \frac{1}{x(x-1)(x+1)}dx$$

Integrating both sides, we get:

$$\int dy = \int \frac{1}{x(x-1)(x+1)} dx \quad \ldots(1)$$

$$\text{Let } \frac{1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}. \quad \ldots(2)$$

$$\Rightarrow \frac{1}{x(x-1)(x+1)} = \frac{A(x-1)(x+1) + Bx(x+1) + Cx(x-1)}{x(x-1)(x+1)}$$

$$= \frac{(A+B+C)x^2 + (B-C)x - A}{x(x-1)(x+1)}$$

Comparing the coefficients of x^2, x, and constant, we get:

$$A = -1$$
$$B - C = 0$$
$$A + B + C = 0$$

Solving these equations, we get

$$B = \frac{1}{2} \text{ and } C = \frac{1}{2}.$$

Substituting the values of A, B, and C in equation (2), we get:

$$\frac{1}{x(x-1)(x+1)} = \frac{-1}{x} + \frac{1}{2(x-1)} + \frac{1}{2(x+1)}$$

Therefore, equation (1) becomes:

$$\int dy = -\int \frac{1}{x}dx + \frac{1}{2}\int \frac{1}{x-1}dx + \frac{1}{2}\int \frac{1}{x+1}dx$$

$$\Rightarrow y = -\log x + \frac{1}{2}\log(x-1) + \frac{1}{2}\log(x+1) + \log k$$

$$\Rightarrow y = \frac{1}{2}\log\left[\frac{k^2(x-1)(x+1)}{x^2}\right] \quad ...(3)$$

Now, $y = 0$ when $x = 2$.

$$\Rightarrow 0 = \frac{1}{2}\log\left[\frac{k^2(2-1)(2+1)}{4}\right]$$

$$\Rightarrow \log\left(\frac{3k^2}{4}\right) = 0$$

$$\Rightarrow \frac{3k^2}{4} = 1$$

$$\Rightarrow 3k^2 = 4$$

$$\Rightarrow k^2 = \frac{4}{3}$$

Substituting the value of k^2 in equation (3), we get:

$$y = \frac{1}{2}\log\left[\frac{4(x-1)(x+1)}{3x^2}\right]$$

$$y = \frac{1}{2}\log\left[\frac{4(x^2-1)}{3x^2}\right]$$

Question 13:

$$\cos\left(\frac{dy}{dx}\right) = a\,(a \in R);\ y = 1 \text{ when } x = 0$$

Answer

$$\cos\left(\frac{dy}{dx}\right)=a$$

$$\Rightarrow \frac{dy}{dx}=\cos^{-1}a$$

$$\Rightarrow dy=\cos^{-1}a\,dx$$

Integrating both sides, we get:

$$\int dy=\cos^{-1}a\int dx$$

$$\Rightarrow y=\cos^{-1}a\cdot x+C$$

$$\Rightarrow y=x\cos^{-1}a+C \qquad \ldots(1)$$

Now, $y=1$ when $x=0$.

$$\Rightarrow 1=0\cdot\cos^{-1}a+C$$

$$\Rightarrow C=1$$

Substituting C = 1 in equation (1), we get:

$$y=x\cos^{-1}a+1$$

$$\Rightarrow \frac{y-1}{x}=\cos^{-1}a$$

$$\Rightarrow \cos\left(\frac{y-1}{x}\right)=a$$

Question 14:

$$\frac{dy}{dx}=y\tan x;\ y=1 \text{ when } x=0$$

Answer

$$\frac{dy}{dx}=y\tan x$$

$$\Rightarrow \frac{dy}{y}=\tan x\,dx$$

Integrating both sides, we get:

$$\int \frac{dy}{y} = -\int \tan x \, dx$$
$$\Rightarrow \log y = \log(\sec x) + \log C$$
$$\Rightarrow \log y = \log(C \sec x)$$
$$\Rightarrow y = C \sec x \qquad ...(1)$$

Now, $y = 1$ when $x = 0$.
$$\Rightarrow 1 = C \times \sec 0$$
$$\Rightarrow 1 = C \times 1$$
$$\Rightarrow C = 1$$

Substituting C = 1 in equation (1), we get:

y = sec x

Question 15:

Find the equation of a curve passing through the point (0, 0) and whose differential equation is $y' = e^x \sin x$.

Answer

The differential equation of the curve is:

$$y' = e^x \sin x$$
$$\Rightarrow \frac{dy}{dx} = e^x \sin x$$
$$\Rightarrow dy = e^x \sin x$$

Integrating both sides, we get:

$$\int dy = \int e^x \sin x \, dx \qquad ...(1)$$

Let $I = \int e^x \sin x \, dx$.

$$\Rightarrow I = \sin x \int e^x dx - \int \left(\frac{d}{dx}(\sin x) \cdot \int e^x dx \right) dx$$

$$\Rightarrow I = \sin x \cdot e^x - \int \cos x \cdot e^x dx$$

$$\Rightarrow I = \sin x \cdot e^x - \left[\cos x \cdot \int e^x dx - \int \left(\frac{d}{dx}(\cos x) \cdot \int e^x dx \right) dx \right]$$

$$\Rightarrow I = \sin x \cdot e^x - \left[\cos x \cdot e^x - \int (-\sin x) \cdot e^x dx \right]$$

$$\Rightarrow I = e^x \sin x - e^x \cos x - I$$

$$\Rightarrow 2I = e^x (\sin x - \cos x)$$

$$\Rightarrow I = \frac{e^x (\sin x - \cos x)}{2}$$

Substituting this value in equation (1), we get:

$$y = \frac{e^x (\sin x - \cos x)}{2} + C \qquad ...(2)$$

Now, the curve passes through point (0, 0).

$$\therefore 0 = \frac{e^0 (\sin 0 - \cos 0)}{2} + C$$

$$\Rightarrow 0 = \frac{1(0-1)}{2} + C$$

$$\Rightarrow C = \frac{1}{2}$$

Substituting $C = \frac{1}{2}$ in equation (2), we get:

$$y = \frac{e^x (\sin x - \cos x)}{2} + \frac{1}{2}$$

$$\Rightarrow 2y = e^x (\sin x - \cos x) + 1$$

$$\Rightarrow 2y - 1 = e^x (\sin x - \cos x)$$

Hence, the required equation of the curve is $2y - 1 = e^x (\sin x - \cos x)$.

Question 16:

For the differential equation $xy\frac{dy}{dx}=(x+2)(y+2),$ find the solution curve passing through the point (1, −1).

Answer

The differential equation of the given curve is:

$$xy\frac{dy}{dx}=(x+2)(y+2)$$

$$\Rightarrow\left(\frac{y}{y+2}\right)dy=\left(\frac{x+2}{x}\right)dx$$

$$\Rightarrow\left(1-\frac{2}{y+2}\right)dy=\left(1+\frac{2}{x}\right)dx$$

Integrating both sides, we get:

$$\int\left(1-\frac{2}{y+2}\right)dy=\int\left(1+\frac{2}{x}\right)dx$$

$$\Rightarrow \int dy-2\int\frac{1}{y+2}dy=\int dx+2\int\frac{1}{x}dx$$

$$\Rightarrow y-2\log(y+2)=x+2\log x+C$$

$$\Rightarrow y-x-C=\log x^2+\log(y+2)^2$$

$$\Rightarrow y-x-C=\log\left[x^2(y+2)^2\right] \quad ...(1)$$

Now, the curve passes through point (1, −1).

$$\Rightarrow -1-1-C=\log\left[(1)^2(-1+2)^2\right]$$

$$\Rightarrow -2-C=\log 1=0$$

$$\Rightarrow C=-2$$

Substituting C = −2 in equation (1), we get:

$$y-x+2=\log\left[x^2(y+2)^2\right]$$

This is the required solution of the given curve.

Question 17:

Find the equation of a curve passing through the point (0, −2) given that at any point (x, y) on the curve, the product of the slope of its tangent and y-coordinate of the point is equal to the x-coordinate of the point.

Answer

Let x and y be the x-coordinate and y-coordinate of the curve respectively.

We know that the slope of a tangent to the curve in the coordinate axis is given by the relation,

$$\frac{dy}{dx}$$

According to the given information, we get:

$$y \cdot \frac{dy}{dx} = x$$

$$\Rightarrow y\,dy = x\,dx$$

Integrating both sides, we get:

$$\int y\,dy = \int x\,dx$$

$$\Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + C$$

$$\Rightarrow y^2 - x^2 = 2C \quad \ldots(1)$$

Now, the curve passes through point (0, −2).

$\therefore (-2)^2 - 0^2 = 2C$

$\Rightarrow 2C = 4$

Substituting 2C = 4 in equation (1), we get:

$y^2 - x^2 = 4$

This is the required equation of the curve.

Question 18:

At any point (x, y) of a curve, the slope of the tangent is twice the slope of the line segment joining the point of contact to the point (−4, −3). Find the equation of the curve given that it passes through (−2, 1).

Answer

It is given that (x, y) is the point of contact of the curve and its tangent.

The slope (m_1) of the line segment joining (x, y) and (−4, −3) is $\frac{y+3}{x+4}$.

We know that the slope of the tangent to the curve is given by the relation,

$$\therefore \text{Slope } (m_2) \text{ of the tangent } = \frac{dy}{dx}$$

According to the given information:

$$m_2 = 2m_1$$
$$\Rightarrow \frac{dy}{dx} = \frac{2(y+3)}{x+4}$$
$$\Rightarrow \frac{dy}{y+3} = \frac{2dx}{x+4}$$

Integrating both sides, we get:

$$\int \frac{dy}{y+3} = 2\int \frac{dx}{x+4}$$
$$\Rightarrow \log(y+3) = 2\log(x+4) + \log C$$
$$\Rightarrow \log(y+3) \log C(x+4)^2$$
$$\Rightarrow y+3 = C(x+4)^2 \quad \ldots(1)$$

This is the general equation of the curve.

It is given that it passes through point (−2, 1).

$$\Rightarrow 1+3 = C(-2+4)^2$$
$$\Rightarrow 4 = 4C$$
$$\Rightarrow C = 1$$

Substituting C = 1 in equation (1), we get:

$y + 3 = (x + 4)^2$

This is the required equation of the curve.

Question 19:

The volume of spherical balloon being inflated changes at a constant rate. If initially its radius is 3 units and after 3 seconds it is 6 units. Find the radius of balloon after t seconds.

Answer

Let the rate of change of the volume of the balloon be k (where k is a constant).

$\Rightarrow \frac{dv}{dt} = k$

$\Rightarrow \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = k \qquad \left[\text{Volume of sphere} = \frac{4}{3}\pi r^3\right]$

$\Rightarrow \frac{4}{3}\pi \cdot 3r^2 \cdot \frac{dr}{dt} = k$

$\Rightarrow 4\pi r^2\, dr = k\, dt$

Integrating both sides, we get:

$4\pi \int r^2 dr = k \int dt$

$\Rightarrow 4\pi \cdot \frac{r^3}{3} = kt + C$

$\Rightarrow 4\pi r^3 = 3(kt + C) \qquad \ldots(1)$

Now, at $t = 0$, $r = 3$:

$\Rightarrow 4\pi \times 3^3 = 3 (k \times 0 + C)$

$\Rightarrow 108\pi = 3C$

$\Rightarrow C = 36\pi$

At t = 3, r = 6:

$\Rightarrow 4\pi \times 6^3 = 3 (k \times 3 + C)$

$\Rightarrow 864\pi = 3 (3k + 36\pi)$

$\Rightarrow 3k = -288\pi - 36\pi = 252\pi$

$\Rightarrow k = 84\pi$

Substituting the values of k and C in equation (1), we get:

$$4\pi r^3 = 3[84\pi t + 36\pi]$$
$$\Rightarrow 4\pi r^3 = 4\pi(63t + 27)$$
$$\Rightarrow r^3 = 63t + 27$$
$$\Rightarrow r = (63t + 27)^{\frac{1}{3}}$$

Thus, the radius of the balloon after t seconds is $(63t+27)^{\frac{1}{3}}$.

Question 20:

In a bank, principal increases continuously at the rate of r% per year. Find the value of r if Rs 100 doubles itself in 10 years ($\log_e 2 = 0.6931$).

Answer

Let p, t, and r represent the principal, time, and rate of interest respectively.

It is given that the principal increases continuously at the rate of r% per year.

$$\Rightarrow \frac{dp}{dt} = \left(\frac{r}{100}\right)p$$
$$\Rightarrow \frac{dp}{p} = \left(\frac{r}{100}\right)dt$$

Integrating both sides, we get:

$$\int \frac{dp}{p} = \frac{r}{100}\int dt$$
$$\Rightarrow \log p = \frac{rt}{100} + k$$
$$\Rightarrow p = e^{\frac{rt}{100}+k} \quad \ldots(1)$$

It is given that when t = 0, p = 100.

$\Rightarrow 100 = e^k \ldots (2)$

Now, if t = 10, then p = 2 × 100 = 200.

Therefore, equation (1) becomes:

$$200 = e^{\frac{r}{10}+k}$$

$$\Rightarrow 200 = e^{\frac{r}{10}} \cdot e^{k}$$

$$\Rightarrow 200 = e^{\frac{r}{10}} \cdot 100 \qquad \text{(From (2))}$$

$$\Rightarrow e^{\frac{r}{10}} = 2$$

$$\Rightarrow \frac{r}{10} = \log_e 2$$

$$\Rightarrow \frac{r}{10} = 0.6931$$

$$\Rightarrow r = 6.931$$

Hence, the value of r is 6.93%.

Question 21:

In a bank, principal increases continuously at the rate of 5% per year. An amount of Rs 1000 is deposited with this bank, how much will it worth after 10 years $\left(e^{0.5} = 1.648\right)$.

Answer

Let p and t be the principal and time respectively.

It is given that the principal increases continuously at the rate of 5% per year.

$$\Rightarrow \frac{dp}{dt} = \left(\frac{5}{100}\right)p$$

$$\Rightarrow \frac{dp}{dt} = \frac{p}{20}$$

$$\Rightarrow \frac{dp}{p} = \frac{dt}{20}$$

Integrating both sides, we get:

$$\int \frac{dp}{p} = \frac{1}{20}\int dt$$

$$\Rightarrow \log p = \frac{t}{20} + C$$

$$\Rightarrow p = e^{\frac{t}{20}+C} \qquad \ldots(1)$$

Now, when t = 0, p = 1000.

⇒ 1000 = e^C … (2)

At t = 10, equation (1) becomes:

$$p = e^{\frac{1}{2}+C}$$

$$\Rightarrow p = e^{0.5} \times e^{C}$$

$$\Rightarrow p = 1.648 \times 1000$$

$$\Rightarrow p = 1648$$

Hence, after 10 years the amount will worth Rs 1648.

Question 22:

In a culture, the bacteria count is 1,00,000. The number is increased by 10% in 2 hours. In how many hours will the count reach 2,00,000, if the rate of growth of bacteria is proportional to the number present?

Answer

Let y be the number of bacteria at any instant t.

It is given that the rate of growth of the bacteria is proportional to the number present.

$$\therefore \frac{dy}{dt} \propto y$$

$$\Rightarrow \frac{dy}{dt} = ky \text{ (where } k \text{ is a constant)}$$

$$\Rightarrow \frac{dy}{y} = kdt$$

Integrating both sides, we get:

$$\int \frac{dy}{y} = k \int dt$$

$$\Rightarrow \log y = kt + C \qquad \ldots(1)$$

Let y_0 be the number of bacteria at t = 0.

⇒ log y_0 = C

Substituting the value of C in equation (1), we get:

$$\log y = kt + \log y_0$$
$$\Rightarrow \log y - \log y_0 = kt$$
$$\Rightarrow \log\left(\frac{y}{y_0}\right) = kt$$
$$\Rightarrow kt = \log\left(\frac{y}{y_0}\right) \qquad ...(2)$$

Also, it is given that the number of bacteria increases by 10% in 2 hours.

$$\Rightarrow y = \frac{110}{100} y_0$$
$$\Rightarrow \frac{y}{y_0} = \frac{11}{10} \qquad ...(3)$$

Substituting this value in equation (2), we get:

$$k \cdot 2 = \log\left(\frac{11}{10}\right)$$
$$\Rightarrow k = \frac{1}{2}\log\left(\frac{11}{10}\right)$$

Therefore, equation (2) becomes:

$$\frac{1}{2}\log\left(\frac{11}{10}\right) \cdot t = \log\left(\frac{y}{y_0}\right)$$
$$\Rightarrow t = \frac{2\log\left(\frac{y}{y_0}\right)}{\log\left(\frac{11}{10}\right)} \qquad ...(4)$$

Now, let the time when the number of bacteria increases from 100000 to 200000 be t_1.

$\Rightarrow y = 2y_0$ at $t = t_1$

From equation (4), we get:

$$t_1 = \frac{2\log\left(\frac{y}{y_0}\right)}{\log\left(\frac{11}{10}\right)} = \frac{2\log 2}{\log\left(\frac{11}{10}\right)}$$

Hence, in $\frac{2\log 2}{\log\left(\frac{11}{10}\right)}$ hours the number of bacteria increases from 100000 to 200000.

Question 23:

The general solution of the differential equation $\frac{dy}{dx} = e^{x+y}$ is

A. $e^x + e^{-y} = C$

B. $e^x + e^y = C$

C. $e^{-x} + e^y = C$

D. $e^{-x} + e^{-y} = C$

Answer

$$\frac{dy}{dx} = e^{x+y} = e^x \cdot e^y$$

$$\Rightarrow \frac{dy}{e^y} = e^x dx$$

$$\Rightarrow e^{-y} dy = e^x dx$$

Integrating both sides, we get:

$$\int e^{-y} dy = \int e^x dx$$

$$\Rightarrow -e^{-y} = e^x + k$$

$$\Rightarrow e^x + e^{-y} = -k$$

$$\Rightarrow e^x + e^{-y} = c \qquad (c = -k)$$

Hence, the correct answer is A.