Exercise 10.4

Question 1:

Find

$|\vec{a}\times\vec{b}|$, if $\vec{a}=\hat{i}-7\hat{j}+7\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+2\hat{k}$.

Answer

We have,

$\vec{a}=\hat{i}-7\hat{j}+7\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+2\hat{k}$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -7 & 7\\ 3 & -2 & 2\end{vmatrix}$$

$$=\hat{i}(-14+14)-\hat{j}(2-21)+\hat{k}(-2+21)=19\hat{j}+19\hat{k}$$

$$\therefore|\vec{a}\times\vec{b}|=\sqrt{(19)^2+(19)^2}=\sqrt{2\times(19)^2}=19\sqrt{2}$$

Question 2:

Find a unit vector perpendicular to each of the vector $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$, where $\vec{a}=3\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}-2\hat{k}$

Answer

We have,

$\vec{a}=3\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}-2\hat{k}$

$$\therefore\vec{a}+\vec{b}=4\hat{i}+4\hat{j},\ \vec{a}-\vec{b}=2\hat{i}+4\hat{k}$$

$$(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & 4 & 0\\ 2 & 0 & 4\end{vmatrix}=\hat{i}(16)-\hat{j}(16)+\hat{k}(-8)=16\hat{i}-16\hat{j}-8\hat{k}$$

$$\therefore|(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})|=\sqrt{16^2+(-16)^2+(-8)^2}$$

$$=\sqrt{2^2\times8^2+2^2\times8^2+8^2}$$

$$=8\sqrt{2^2+2^2+1}=8\sqrt{9}=8\times3=24$$

Hence, the unit vector perpendicular to each of the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ is given by the relation,

$$=\pm\frac{(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})}{\left|(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})\right|}=\pm\frac{16\hat{i}-16\hat{j}-8\hat{k}}{24}$$

$$=\pm\frac{2\hat{i}-2\hat{j}-\hat{k}}{3}=\pm\frac{2}{3}\hat{i}\mp\frac{2}{3}\hat{j}\mp\frac{1}{3}\hat{k}$$

Question 3:

If a unit vector $\vec{a}$ makes an $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ angle with $\hat{j}$ and an acute angle θ with $\hat{k}$, then find θ and hence, the compounds of $\vec{a}$

Answer

Let unit vector $\vec{a}$ have (a_1, a_2, a_3) components.

$$\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$$

Since $\vec{a}$ is a unit vector, $|\vec{a}|=1.$

Also, it is given that $\vec{a}$ makes angles $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$, and an acute angle θ with $\hat{k}$. Then, we have:

$$\cos\frac{\pi}{3}=\frac{a_1}{|\vec{a}|}$$

$$\Rightarrow \frac{1}{2}=a_1 \qquad \left[|\vec{a}|=1\right]$$

$$\cos\frac{\pi}{4}=\frac{a_2}{|\vec{a}|}$$

$$\Rightarrow \frac{1}{\sqrt{2}}=a_2 \qquad \left[|\vec{a}|=1\right]$$

Also, $\cos\theta=\frac{a_3}{|\vec{a}|}$.

$\Rightarrow a_3=\cos\theta$

Now,

$|a|=1$

$$\Rightarrow \sqrt{a_1^2+a_2^2+a_3^2}=1$$

$$\Rightarrow \left(\frac{1}{2}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2+\cos^2\theta=1$$

$$\Rightarrow \frac{1}{4}+\frac{1}{2}+\cos^2\theta=1$$

$$\Rightarrow \frac{3}{4}+\cos^2\theta=1$$

$$\Rightarrow \cos^2\theta=1-\frac{3}{4}=\frac{1}{4}$$

$$\Rightarrow \cos\theta=\frac{1}{2}\Rightarrow\theta=\frac{\pi}{3}$$

$$\therefore a_3=\cos\frac{\pi}{3}=\frac{1}{2}$$

Hence, $\theta=\frac{\pi}{3}$ and the components of $\vec{a}$ are $\left(\frac{1}{2},\frac{1}{\sqrt{2}},\frac{1}{2}\right)$

Question 4:

Show that

$$\left(\vec{a}-\vec{b}\right)\times\left(\vec{a}+\vec{b}\right)=2\left(\vec{a}\times\vec{b}\right)$$

Answer

$$\left(\vec{a}-\vec{b}\right)\times\left(\vec{a}+\vec{b}\right)$$

$$=\left(\vec{a}-\vec{b}\right)\times\vec{a}+\left(\vec{a}-\vec{b}\right)\times\vec{b} \qquad \text{[By distributivity of vector product over addition]}$$

$$=\vec{a}\times\vec{a}-\vec{b}\times\vec{a}+\vec{a}\times\vec{b}-\vec{b}\times\vec{b} \qquad \text{[Again, by distributivity of vector product over addition]}$$

$$=\vec{0}+\vec{a}\times\vec{b}+\vec{a}\times\vec{b}-\vec{0}$$

$$=2\vec{a}\times\vec{b}$$

Question 5:

Find λ and μ if $\left(2\hat{i}+6\hat{j}+27\hat{k}\right)\times\left(\hat{i}+\lambda\hat{j}+\mu\hat{k}\right)=\vec{0}$.

Answer

$$\left(2\hat{i}+6\hat{j}+27\hat{k}\right)\times\left(\hat{i}+\lambda\hat{j}+\mu\hat{k}\right)=\vec{0}$$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 6 & 27 \\ 1 & \lambda & \mu \end{vmatrix} = 0\hat{i}+0\hat{j}+0\hat{k}$$

$$\Rightarrow \hat{i}\left(6\mu-27\lambda\right)-\hat{j}\left(2\mu-27\right)+\hat{k}\left(2\lambda-6\right)=0\hat{i}+0\hat{j}+0\hat{k}$$

On comparing the corresponding components, we have:

$6\mu - 27\lambda = 0$
$2\mu - 27 = 0$
$2\lambda - 6 = 0$

Now,
$2\lambda - 6 = 0 \Rightarrow \lambda = 3$
$2\mu - 27 = 0 \Rightarrow \mu = \frac{27}{2}$

Hence, $\lambda = 3$ and $\mu = \frac{27}{2}$.

Question 6:
Given that
$\vec{a} \cdot \vec{b} = 0$ and $\vec{a} \times \vec{b} = \vec{0}$.

What can you conclude about the vectors $\vec{a}$ and $\vec{b}$?

Answer
$\vec{a} \cdot \vec{b} = 0$
Then,
(i) Either $|\vec{a}| = 0$ or $|\vec{b}| = 0$, or $\vec{a} \perp \vec{b}$ (in case $\vec{a}$ and $\vec{b}$ are non-zero)
$\vec{a} \times \vec{b} = 0$

(ii) Either $|\vec{a}| = 0$ or $|\vec{b}| = 0$, or $\vec{a} \parallel \vec{b}$ (in case $\vec{a}$ and $\vec{b}$ are non-zero)

But, $\vec{a}$ and $\vec{b}$ cannot be perpendicular and parallel simultaneously.

Hence, $|\vec{a}| = 0$ or $|\vec{b}| = 0$

Question 7:

Let the vectors $\vec{a},\ \vec{b},\ \vec{c}$ given as $a_1\hat{i}+a_2\hat{j}+a_3\hat{k},\ b_1\hat{i}+b_2\hat{j}+b_3\hat{k},\ \ c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$

Then show that $=\vec{a}\times\left(\vec{b}+\vec{c}\right)=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$

Answer

We have,

$$\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k},\ \vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k},\ \vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$$

$$\left(\vec{b}+\vec{c}\right)=\left(b_1+c_1\right)\hat{i}+\left(b_2+c_2\right)\hat{j}+\left(b_3+c_3\right)\hat{k}$$

$$\text{Now, } \vec{a}\times\left(\vec{b}+\vec{c}\right)\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1+c_1 & b_2+c_2 & b_3+c_3\end{vmatrix}$$

$$=\hat{i}\left[a_2\left(b_3+c_3\right)-a_3\left(b_2+c_2\right)\right]-\hat{j}\left[a_1\left(b_3+c_3\right)-a_3\left(b_1+c_1\right)\right]+\hat{k}\left[a_1\left(b_2+c_2\right)-a_2\left(b_1+c_1\right)\right]$$

$$=\hat{i}\left[a_2b_3+a_2c_3-a_3b_2-a_3c_2\right]+\hat{j}\left[-a_1b_3-a_1c_3+a_3b_1+a_3c_1\right]+\hat{k}\left[a_1b_2+a_1c_2-a_2b_1-a_2c_1\right]\quad ...(1)$$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

$$=\hat{i}\left[a_2b_3-a_3b_2\right]+\hat{j}\left[b_1a_3-a_1b_3\right]+\hat{k}\left[a_1b_2-a_2b_1\right]\qquad(2)$$

$$\vec{a}\times\vec{c}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ c_1 & c_2 & c_3\end{vmatrix}$$

$$=\hat{i}\left[a_2c_3-a_3c_2\right]+\hat{j}\left[a_3c_1-a_1c_3\right]+\hat{k}\left[a_1c_2-a_2c_1\right]\qquad(3)$$

On adding (2) and (3), we get:

$$\left(\vec{a}\times\vec{b}\right)+\left(\vec{a}\times\vec{c}\right)=\hat{i}\left[a_2b_3+a_2c_3-a_3b_2-a_3c_2\right]+\hat{j}\left[b_1a_3+a_3c_1-a_1b_3-a_1c_3\right]$$
$$+\hat{k}\left[a_1b_2+a_1c_2-a_2b_1-a_2c_1\right]\qquad(4)$$

Now, from (1) and (4), we have:

$$\vec{a}\times\left(\vec{b}+\vec{c}\right)=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$$

Hence, the given result is proved.

Question 8:

If either $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$, then $\vec{a}\times\vec{b}=\vec{0}$.

Is the converse true? Justify your answer with an example.

Answer

Take any parallel non-zero vectors so that $\vec{a}\times\vec{b}=\vec{0}$.

Let $\vec{a}=2\hat{i}+3\hat{j}+4\hat{k},\ \vec{b}=4\hat{i}+6\hat{j}+8\hat{k}$.

Then,

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & 4\\ 4 & 6 & 8\end{vmatrix}=\hat{i}(24-24)-\hat{j}(16-16)+\hat{k}(12-12)=0\hat{i}+0\hat{j}+0\hat{k}=\vec{0}$$

It can now be observed that:

$$\left|\vec{a}\right|=\sqrt{2^2+3^2+4^2}=\sqrt{29}$$

$$\therefore \vec{a}\neq\vec{0}$$

$$\left|\vec{b}\right|=\sqrt{4^2+6^2+8^2}=\sqrt{116}$$

$$\therefore \vec{b}\neq\vec{0}$$

Hence, the converse of the given statement need not be true.

Question 9:

Find the area of the triangle with vertices A (1, 1, 2), B (2, 3, 5) and C (1, 5, 5).

Answer

The vertices of triangle ABC are given as A (1, 1, 2), B (2, 3, 5), and C (1, 5, 5).

The adjacent sides $\overrightarrow{AB}$ and $\overrightarrow{BC}$ of ΔABC are given as:

$$\overrightarrow{AB}=(2-1)\hat{i}+(3-1)\hat{j}+(5-2)\hat{k}=\hat{i}+2\hat{j}+3\hat{k}$$

$$\overrightarrow{BC}=(1-2)\hat{i}+(5-3)\hat{j}+(5-5)\hat{k}=-\hat{i}+2\hat{j}$$

Area of ΔABC $=\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{BC}\right|$

$$\overrightarrow{AB}\times\overrightarrow{BC}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & 3\\ -1 & 2 & 0\end{vmatrix}=\hat{i}(-6)-\hat{j}(3)+\hat{k}(2+2)=-6\hat{i}-3\hat{j}+4\hat{k}$$

$$\therefore\left|\overrightarrow{AB}\times\overrightarrow{BC}\right|=\sqrt{(-6)^2+(-3)^2+4^2}=\sqrt{36+9+16}=\sqrt{61}$$

Hence, the area of ΔABC is $\frac{\sqrt{61}}{2}$ square units.

Question 10:

Find the area of the parallelogram whose adjacent sides are determined by the vector $\vec{a}=\hat{i}-\hat{j}+3\hat{k}$ and $\vec{b}=2\hat{i}-7\hat{j}+\hat{k}$.

Answer

The area of the parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$ is $\left|\vec{a}\times\vec{b}\right|$.

Adjacent sides are given as:

$\vec{a}=\hat{i}-\hat{j}+3\hat{k}$ and $\vec{b}=2\hat{i}-7\hat{j}+\hat{k}$

$$\therefore\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & 3\\ 2 & -7 & 1\end{vmatrix}=\hat{i}(-1+21)-\hat{j}(1-6)+\hat{k}(-7+2)=20\hat{i}+5\hat{j}-5\hat{k}$$

$$\left|\vec{a}\times\vec{b}\right|=\sqrt{20^2+5^2+5^2}=\sqrt{400+25+25}=15\sqrt{2}$$

Hence, the area of the given parallelogram is $15\sqrt{2}$ square units

Question 11:

Let the vectors $\vec{a}$ and $\vec{b}$ be such that $|\vec{a}| = 3$ and $|\vec{b}| = \frac{\sqrt{2}}{3}$, then $\vec{a} \times \vec{b}$ is a unit vector, if the angle between $\vec{a}$ and $\vec{b}$ is

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

Answer

It is given that $|\vec{a}| = 3$ and $|\vec{b}| = \frac{\sqrt{2}}{3}$

We know that $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$, where is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ and θ is the angle between $\vec{a}$ and $\vec{b}$.

Now, $\vec{a} \times \vec{b}$ is a unit vector if $|\vec{a} \times \vec{b}| = 1$

$$|\vec{a} \times \vec{b}| = 1$$

$$\Rightarrow \left||\vec{a}||\vec{b}|\sin\theta\,\hat{n}\right| = 1$$

$$\Rightarrow |\vec{a}||\vec{b}||\sin\theta| = 1$$

$$\Rightarrow 3 \times \frac{\sqrt{2}}{3} \times \sin\theta = 1$$

$$\Rightarrow \sin\theta = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \theta = \frac{\pi}{4}$$

Hence, $\vec{a} \times \vec{b}$ is a unit vector if the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{4}$.

The correct answer is B.

Question 12:

Area of a rectangle having vertices A, B, C, and D with position vectors

$-\hat{i}+\frac{1}{2}\hat{j}+4\hat{k},\ \hat{i}+\frac{1}{2}\hat{j}+4\hat{k},\ \hat{i}-\frac{1}{2}\hat{j}+4\hat{k}$ and $-\hat{i}-\frac{1}{2}\hat{j}+4\hat{k}$ respectively is

(A) $\frac{1}{2}$ (B) 1

(C) 2 (D) 4

Answer

The position vectors of vertices A, B, C, and D of rectangle ABCD are given as:

$$\overrightarrow{OA}=-\hat{i}+\frac{1}{2}\hat{j}+4\hat{k},\ \overrightarrow{OB}=\hat{i}+\frac{1}{2}\hat{j}+4\hat{k},\ \overrightarrow{OC}=\hat{i}-\frac{1}{2}\hat{j}+4\hat{k},\ \overrightarrow{OD}=-\hat{i}-\frac{1}{2}\hat{j}+4\hat{k}$$

The adjacent sides $\overrightarrow{AB}$ and $\overrightarrow{BC}$ of the given rectangle are given as:

$$\overrightarrow{AB}=(1+1)\hat{i}+\left(\frac{1}{2}-\frac{1}{2}\right)\hat{j}+(4-4)\hat{k}=2\hat{i}$$

$$\overrightarrow{BC}=(1-1)\hat{i}+\left(-\frac{1}{2}-\frac{1}{2}\right)\hat{j}+(4-4)\hat{k}=-\hat{j}$$

$$\therefore \overrightarrow{AB}\times\overrightarrow{BC}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 0 & 0\\ 0 & -1 & 0\end{vmatrix}=\hat{k}(-2)=-2\hat{k}$$

$$\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\sqrt{(-2)^2}=2$$

Now, it is known that the area of a parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$ is $\left|\vec{a}\times\vec{b}\right|$.

Hence, the area of the given rectangle is $\left|\overrightarrow{AB}\times\overrightarrow{BC}\right|=2$ square units.

The correct answer is C.