Miscellaneous Solutions

Question 1:

Show that the line joining the origin to the point (2, 1, 1) is perpendicular to the line determined by the points (3, 5, −1), (4, 3, −1).

Answer

Let OA be the line joining the origin, O (0, 0, 0), and the point, A (2, 1, 1).

Also, let BC be the line joining the points, B (3, 5, −1) and C (4, 3, −1).

The direction ratios of OA are 2, 1, and 1 and of BC are (4 − 3) = 1, (3 − 5) = −2, and (−1 + 1) = 0

OA is perpendicular to BC, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

$\therefore a_1a_2 + b_1b_2 + c_1c_2 = 2 \times 1 + 1(-2) + 1 \times 0 = 2 - 2 = 0$

Thus, OA is perpendicular to BC.

Question 2:

If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two mutually perpendicular lines, show that the direction cosines of the line perpendicular to both of these are $m_1n_2 - m_2n_1$, $n_1l_2 - n_2l_1$, $l_1m_2 - l_2m_1$.

Answer

It is given that l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two mutually perpendicular lines. Therefore,

$$l_1l_2 + m_1m_2 + n_1n_2 = 0 \qquad ...(1)$$

$$l_1^2 + m_1^2 + n_1^2 = 1 \qquad ...(2)$$

$$l_2^2 + m_2^2 + n_2^2 = 1 \qquad ...(3)$$

Let l, m, n be the direction cosines of the line which is perpendicular to the line with direction cosines l_1, m_1, n_1 and l_2, m_2, n_2.

$$\therefore ll_1 + mm_1 + nn_1 = 0$$

$$ll_2 + mm_2 + nn_2 = 0$$

$$\therefore \frac{l}{m_1n_2 - m_2n_1} = \frac{m}{n_1l_2 - n_2l_1} = \frac{n}{l_1m_2 - l_2m_1}$$

$$\Rightarrow \frac{l^2}{(m_1n_2 - m_2n_1)^2} = \frac{m^2}{(n_1l_2 - n_2l_1)^2} = \frac{n^2}{(l_1m_2 - l_2m_1)^2}$$

$$\Rightarrow \frac{l^2}{(m_1n_2 - m_2n_1)^2} = \frac{m^2}{(n_1l_2 - n_2l_1)^2} = \frac{n^2}{(l_1m_2 - l_2m_1)^2}$$

$$= \frac{l^2 + m^2 + n^2}{(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2} \qquad ...(4)$$

l, m, n are the direction cosines of the line. $\therefore l^2 + m^2 + n^2 = 1$ ... (5)

It is known that,

$$(l_1^2 + m_1^2 + n_1^2)(l_2^2 + m_2^2 + n_2^2) - (l_1l_2 + m_1m_2 + n_1n_2)^2$$
$$= (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2$$

From (1), (2), and (3), we obtain

$$\Rightarrow 1.1 - 0 = (m_1n_2 + m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2$$

$$\therefore (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2 = 1 \qquad ...(6)$$

Substituting the values from equations (5) and (6) in equation (4), we obtain

$$\frac{l^2}{(m_1n_2 - m_2n_1)^2} = \frac{m^2}{(n_2l_2 - n_2l_1)^2} = \frac{n^2}{(l_1m_2 - l_2m_1)^2} = 1$$

$$\Rightarrow l = m_1n_2 - m_2n_1, m = n_1l_2 - n_2l_1, n = l_1m_2 - l_2m_1$$

Thus, the direction cosines of the required line are $m_1n_2 - m_2n_1,\ n_1l_2 - n_2l_1, \text{and}\ l_1m_2 - l_2m_1.$

Question 3:

Find the angle between the lines whose direction ratios are a, b, c and b − c, c − a, a − b.

Answer

The angle Q between the lines with direction cosines, a, b, c and b − c, c − a, a − b, is given by,

$$\cos Q=\left|\frac{a(b-c)+b(c-a)+c(a-b)}{\sqrt{a^2+b^2+c^2}+\sqrt{(b-c)^2+(c-a)^2+(a-b)^2}}\right|$$

$$\Rightarrow \cos Q=0$$

$$\Rightarrow Q=\cos^{-1}0$$

$$\Rightarrow Q=90°$$

Thus, the angle between the lines is 90°.

Question 4:

Find the equation of a line parallel to x-axis and passing through the origin.

Answer

The line parallel to x-axis and passing through the origin is x-axis itself.

Let A be a point on x-axis. Therefore, the coordinates of A are given by (a, 0, 0), where a ∈ R.

Direction ratios of OA are (a − 0) = a, 0, 0

The equation of OA is given by,

$$\frac{x-0}{a}=\frac{y-0}{0}=\frac{z-0}{0}$$

$$\Rightarrow \frac{x}{1}=\frac{y}{0}=\frac{z}{0}=a$$

Thus, the equation of line parallel to x-axis and passing through origin is

$$\frac{x}{1}=\frac{y}{0}=\frac{z}{0}$$

Question 5:

If the coordinates of the points A, B, C, D be (1, 2, 3), (4, 5, 7), (−4, 3, −6) and (2, 9, 2) respectively, then find the angle between the lines AB and CD.

Answer

The coordinates of A, B, C, and D are (1, 2, 3), (4, 5, 7), (−4, 3, −6), and (2, 9, 2) respectively.

The direction ratios of AB are (4 − 1) = 3, (5 − 2) = 3, and (7 − 3) = 4

The direction ratios of CD are (2 −(− 4)) = 6, (9 − 3) = 6, and (2 −(−6)) = 8

It can be seen that, $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}=\frac{1}{2}$

Therefore, AB is parallel to CD.

Thus, the angle between AB and CD is either 0° or 180°.

Question 6:

If the lines $\frac{x-1}{-3}=\frac{y-2}{2k}=\frac{z-3}{2}$ and $\frac{x-1}{3k}=\frac{y-1}{1}=\frac{z-6}{-5}$ are perpendicular, find the value of k.

Answer

The direction of ratios of the lines, $\frac{x-1}{-3}=\frac{y-2}{2k}=\frac{z-3}{2}$ and $\frac{x-1}{3k}=\frac{y-1}{1}=\frac{z-6}{-5}$, are −3, 2k, 2 and 3k, 1, −5 respectively.

It is known that two lines with direction ratios, a_1, b_1, c_1 and a_2, b_2, c_2, are perpendicular, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

$$\therefore -3(3k)+2k\times 1+2(-5)=0$$
$$\Rightarrow -9k+2k-10=0$$
$$\Rightarrow 7k=-10$$
$$\Rightarrow k=\frac{-10}{7}$$

Therefore, for $k=-\frac{10}{7}$, the given lines are perpendicular to each other.

Question 7:

Find the vector equation of the plane passing through (1, 2, 3) and perpendicular to the plane $\vec{r}\cdot\left(\hat{i}+2\hat{j}-5\hat{k}\right)+9=0$

Answer

The position vector of the point (1, 2, 3) is $\vec{r_1}=\hat{i}+2\hat{j}+3\hat{k}$

The direction ratios of the normal to the $\vec{r}\cdot\left(\hat{i}+2\hat{j}-5\hat{k}\right)+9=0$ plane, are 1, 2, and −5

and the normal vector is $\vec{N}=\hat{i}+2\hat{j}-5\hat{k}$

The equation of a line passing through a point and perpendicular to the given plane is given by, $\vec{l}=\vec{r}+\lambda\vec{N},\ \lambda\in R$

$\Rightarrow\vec{l}=\left(\hat{i}+2\hat{j}+3\hat{k}\right)+\lambda\left(\hat{i}+2\hat{j}-5\hat{k}\right)$

Question 8:

Find the equation of the plane passing through (a, b, c) and parallel to the plane $\vec{r}\cdot\left(\hat{i}+\hat{j}+\hat{k}\right)=2$

Answer

Any plane parallel to the plane, $\vec{r_1}\cdot\left(\hat{i}+\hat{j}+\hat{k}\right)=2$

$\vec{r}\cdot\left(\hat{i}+\hat{j}+\hat{k}\right)=\lambda \qquad ...(1)$

The plane passes through the point (a, b, c). Therefore, the position vector $\vec{r}$ of this point is $\vec{r}=a\hat{i}+b\hat{j}+c\hat{k}$

Therefore, equation (1) becomes

$\left(a\hat{i}+b\hat{j}+c\hat{k}\right)\cdot\left(\hat{i}+\hat{j}+\hat{k}\right)=\lambda$

$\Rightarrow a+b+c=\lambda$

Substituting $\lambda=a+b+c$ in equation (1), we obtain

$\vec{r}\cdot\left(\hat{i}+\hat{j}+\hat{k}\right)=a+b+c \qquad ...(2)$

This is the vector equation of the required plane.

Substituting $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ in equation (2), we obtain

$$\left(x\hat{i} + y\hat{j} + z\hat{k}\right)\cdot\left(\hat{i} + \hat{j} + \hat{k}\right) = a + b + c$$

$$\Rightarrow x + y + z = a + b + c$$

Question 9:

Find the shortest distance between lines $\vec{r} = 6\hat{i} + 2\hat{j} + 2\hat{k} + \lambda\left(\hat{i} - 2\hat{j} + 2\hat{k}\right)$ and $\vec{r} = -4\hat{i} - \hat{k} + \mu\left(3\hat{i} - 2\hat{j} - 2\hat{k}\right)$.

Answer

The given lines are

$$\vec{r} = 6\hat{i} + 2\hat{j} + 2\hat{k} + \lambda\left(\hat{i} - 2\hat{j} + 2\hat{k}\right) \quad ...(1)$$

$$\vec{r} = -4\hat{i} - \hat{k} + \mu\left(3\hat{i} - 2\hat{j} - 2\hat{k}\right) \quad ...(2)$$

It is known that the shortest distance between two lines, $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\lambda\vec{b}_2$, is given by

$$d=\left|\frac{\left(\vec{b}_1\times\vec{b}_2\right)\cdot\left(\vec{a}_2-\vec{a}_1\right)}{\left|\vec{b}_1\times\vec{b}_2\right|}\right| \qquad ...(3)$$

Comparing $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\lambda\vec{b}_2$ to equations (1) and (2), we obtain

$$\vec{a}_1=6\hat{i}+2\hat{j}+2\hat{k}$$
$$\vec{b}_1=\hat{i}-2\hat{j}+2\hat{k}$$
$$\vec{a}_2=-4\hat{i}-\hat{k}$$
$$\vec{b}_2=3\hat{i}-2\hat{j}-2\hat{k}$$

$$\Rightarrow \vec{a}_2-\vec{a}_1=\left(-4\hat{i}-\hat{k}\right)-\left(6\hat{i}+2\hat{j}+2\hat{k}\right)=-10\hat{i}-2\hat{j}-3\hat{k}$$

$$\Rightarrow \vec{b}_1\times\vec{b}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -2 & 2\\ 3 & -2 & -2\end{vmatrix}=(4+4)\hat{i}-(-2-6)\hat{j}+(-2+6)\hat{k}=8\hat{i}+8\hat{j}+4\hat{k}$$

$$\therefore \left|\vec{b}_1\times\vec{b}_2\right|=\sqrt{(8)^2+(8)^2+(4)^2}=12$$

$$\left(\vec{b}_1\times\vec{b}_2\right)\cdot\left(\vec{a}_2-\vec{a}_1\right)=\left(8\hat{i}+8\hat{j}+4\hat{k}\right)\cdot\left(-10\hat{i}-2\hat{j}-3\hat{k}\right)=-80-16-12=-108$$

Substituting all the values in equation (1), we obtain

$$d=\left|\frac{-108}{12}\right|=9$$

Therefore, the shortest distance between the two given lines is 9 units.

Question 10:

Find the coordinates of the point where the line through (5, 1, 6) and (3, 4, 1) crosses the YZ-plane

Answer

It is known that the equation of the line passing through the points, (x_1, y_1, z_1) and $(x_2,$

y_2, z_2), is $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$

The line passing through the points, (5, 1, 6) and (3, 4, 1), is given by,

$$\frac{x-5}{3-5}=\frac{y-1}{4-1}=\frac{z-6}{1-6}$$

$$\Rightarrow \frac{x-5}{-2}=\frac{y-1}{3}=\frac{z-6}{-5}=k\left(\text{say}\right)$$

$$\Rightarrow x=5-2k,\ y=3k+1,\ z=6-5k$$

Any point on the line is of the form (5 − 2k, 3k + 1, 6 −5k).

The equation of YZ-plane is x = 0

Since the line passes through YZ-plane,

5 − 2k = 0

$$\Rightarrow k=\frac{5}{2}$$

$$\Rightarrow 3k+1=3\times\frac{5}{2}+1=\frac{17}{2}$$

$$6-5k=6-5\times\frac{5}{2}=\frac{-13}{2}$$

Therefore, the required point is $\left(0,\frac{17}{2},\frac{-13}{2}\right)$.

Question 11:

Find the coordinates of the point where the line through (5, 1, 6) and (3, 4, 1) crosses the ZX − plane.

Answer

It is known that the equation of the line passing through the points, (x_1, y_1, z_1) and (x_2, y_2, z_2), is $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$

The line passing through the points, (5, 1, 6) and (3, 4, 1), is given by,

$$\frac{x-5}{3-5}=\frac{y-1}{4-1}=\frac{z-6}{1-6}$$
$$\Rightarrow \frac{x-5}{-2}=\frac{y-1}{3}=\frac{z-6}{-5}=k\left(\text{say}\right)$$
$$\Rightarrow x=5-2k,\ y=3k+1,\ z=6-5k$$

Any point on the line is of the form (5 − 2k, 3k + 1, 6 −5k).

Since the line passes through ZX-plane,

$$3k+1=0$$
$$\Rightarrow k=-\frac{1}{3}$$
$$\Rightarrow 5-2k=5-2\left(-\frac{1}{3}\right)=\frac{17}{3}$$
$$6-5k=6-5\left(-\frac{1}{3}\right)=\frac{23}{3}$$

Therefore, the required point is $\left(\frac{17}{3},0,\frac{23}{3}\right)$

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Question 12:

Find the coordinates of the point where the line through (3, −4, −5) and (2, − 3, 1) crosses the plane 2x + y + z = 7).

Answer

It is known that the equation of the line through the points, (x_1, y_1, z_1) and (x_2, y_2, z_2), is

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

Since the line passes through the points, (3, −4, −5) and (2, −3, 1), its equation is given by,

$$\frac{x-3}{2-3}=\frac{y+4}{-3+4}=\frac{z+5}{1+5}$$
$$\Rightarrow \frac{x-3}{-1}=\frac{y+4}{1}=\frac{z+5}{6}=k\left(\text{say}\right)$$
$$\Rightarrow x=3-k,\ y=k-4,\ z=6k-5$$

Therefore, any point on the line is of the form (3 − k, k − 4, 6k − 5).

This point lies on the plane, 2x + y + z = 7

∴ 2 (3 − k) + (k − 4) + (6k − 5) = 7

$\Rightarrow 5k-3=7$

$\Rightarrow k=2$

Hence, the coordinates of the required point are (3 − 2, 2 − 4, 6 × 2 − 5) i.e., (1, −2, 7).

Question 13:

Find the equation of the plane passing through the point (−1, 3, 2) and perpendicular to each of the planes x + 2y + 3z = 5 and 3x + 3y + z = 0.

Answer

The equation of the plane passing through the point (−1, 3, 2) is a (x + 1) + b (y − 3) + c (z − 2) = 0 … (1) where, a, b, c are the direction ratios of normal to the plane.

It is known that two planes, $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$, are perpendicular, if $a_1a_2+b_1b_2+c_1c_2=0$

Plane (1) is perpendicular to the plane, x + 2y + 3z = 5

$\therefore a\cdot 1+b\cdot 2+c\cdot 3=0$

$\Rightarrow a+2b+3c=0 \quad \text{...(2)}$

Also, plane (1) is perpendicular to the plane, 3x + 3y + z = 0

$\therefore a\cdot 3+b\cdot 3+c\cdot 1=0$

$\Rightarrow 3a+3b+c=0 \quad \text{...(3)}$

From equations (2) and (3), we obtain

$$\frac{a}{2\times 1-3\times 3}=\frac{b}{3\times 3-1\times 1}=\frac{c}{1\times 3-2\times 3}$$

$$\Rightarrow \frac{a}{-7}=\frac{b}{8}=\frac{c}{-3}=k\left(\text{say}\right)$$

$$\Rightarrow a=-7k,\ b=8k,\ c=-3k$$

Substituting the values of a, b, and c in equation (1), we obtain

$$-7k(x+1)+8k(y-3)-3k(z-2)=0$$
$$\Rightarrow(-7x-7)+(8y-24)-3z+6=0$$
$$\Rightarrow -7x+8y-3z-25=0$$
$$\Rightarrow 7x-8y+3z+25=0$$

This is the required equation of the plane.

Question 14:

If the points (1, 1, p) and (−3, 0, 1) be equidistant from the plane $\vec{r}\cdot\left(3\hat{i}+4\hat{j}-12\hat{k}\right)+13=0$, then find the value of p.

Answer

The position vector through the point (1, 1, p) is $\vec{a}_1=\hat{i}+\hat{j}+p\hat{k}$

Similarly, the position vector through the point (−3, 0, 1) is

$\vec{a}_2=-4\hat{i}+\hat{k}$

The equation of the given plane is $\vec{r}\cdot\left(3\hat{i}+4\hat{j}-12\hat{k}\right)+13=0$

It is known that the perpendicular distance between a point whose position vector is $\vec{a}$ and the plane, $\vec{r}\cdot\vec{N}=d,$ is given by, $D=\dfrac{\left|\vec{a}\cdot\vec{N}-d\right|}{\left|\vec{N}\right|}$

Here, $\vec{N}=3\hat{i}+4\hat{j}-12\hat{k}$ and d $=-13$

Therefore, the distance between the point (1, 1, p) and the given plane is

$$D_1=\frac{\left|\left(\hat{i}+\hat{j}+p\hat{k}\right)\cdot\left(3\hat{i}+4\hat{j}-12\hat{k}\right)+13\right|}{\left|3\hat{i}+4\hat{j}-12\hat{k}\right|}$$
$$\Rightarrow D_1=\frac{\left|3+4-12p+13\right|}{\sqrt{3^2+4^2+(-12)^2}}$$
$$\Rightarrow D_1=\frac{\left|20-12p\right|}{13} \qquad \ldots(1)$$

Similarly, the distance between the point (−3, 0, 1) and the given plane is

$$D_2 = \frac{\left|\left(-3\hat{i}+\hat{k}\right)\cdot\left(3\hat{i}+4\hat{j}-12\hat{k}\right)+13\right|}{\left|3\hat{i}+4\hat{j}-12\hat{k}\right|}$$

$$\Rightarrow D_2 = \frac{\left|-9-12+13\right|}{\sqrt{3^2+4^2+(-12)^2}}$$

$$\Rightarrow D_2 = \frac{8}{13} \qquad \ldots(2)$$

It is given that the distance between the required plane and the points, (1, 1, p) and (−3, 0, 1), is equal.

∴ $D_1 = D_2$

$$\Rightarrow \frac{\left|20-12p\right|}{13} = \frac{8}{13}$$

$$\Rightarrow 20-12p = 8 \text{ or } -(20-12p) = 8$$

$$\Rightarrow 12p = 12 \text{ or } 12p = 28$$

$$\Rightarrow p = 1 \text{ or } p = \frac{7}{3}$$

Question 15:

Find the equation of the plane passing through the line of intersection of the planes

$\vec{r}\cdot\left(\hat{i}+\hat{j}+\hat{k}\right)=1$ and $\vec{r}\cdot\left(2\hat{i}+3\hat{j}-\hat{k}\right)+4=0$

and parallel to x-axis.

Answer

The given planes are

$$\vec{r}\cdot\left(\hat{i}+\hat{j}+\hat{k}\right)=1$$

$$\Rightarrow \vec{r}\cdot\left(\hat{i}+\hat{j}+\hat{k}\right)-1=0$$

$$\vec{r}\cdot\left(2\hat{i}+3\hat{j}-\hat{k}\right)+4=0$$

The equation of any plane passing through the line of intersection of these planes is

$$\left[\vec{r}\cdot\left(\hat{i}+\hat{j}+\hat{k}\right)-1\right]+\lambda\left[\vec{r}\cdot\left(2\hat{i}+3\hat{j}-\hat{k}\right)+4\right]=0$$

$$\vec{r}\cdot\left[(2\lambda+1)\hat{i}+(3\lambda+1)\hat{j}+(1-\lambda)\hat{k}\right]+(4\lambda+1)=0 \qquad ...(1)$$

Its direction ratios are (2λ + 1), (3λ + 1), and (1 − λ).

The required plane is parallel to x-axis. Therefore, its normal is perpendicular to x-axis.

The direction ratios of x-axis are 1, 0, and 0.

$$\therefore 1.(2\lambda+1)+0(3\lambda+1)+0(1-\lambda)=0$$

$$\Rightarrow 2\lambda+1=0$$

$$\Rightarrow \lambda=-\frac{1}{2}$$

Substituting $\lambda=-\frac{1}{2}$ in equation (1), we obtain

$$\Rightarrow \vec{r}\cdot\left[-\frac{1}{2}\hat{j}+\frac{3}{2}\hat{k}\right]+(-3)=0$$

$$\Rightarrow \vec{r}\left(\hat{j}-3\hat{k}\right)+6=0$$

Therefore, its Cartesian equation is y − 3z + 6 = 0

This is the equation of the required plane.

Question 16:

If O be the origin and the coordinates of P be (1, 2, −3), then find the equation of the plane passing through P and perpendicular to OP.

Answer

The coordinates of the points, O and P, are (0, 0, 0) and (1, 2, −3) respectively.

Therefore, the direction ratios of OP are (1 − 0) = 1, (2 − 0) = 2, and (−3 − 0) = −3

It is known that the equation of the plane passing through the point (x_1, y_1 z_1) is

$a(x-x_1)+b(y-y_1)+c(z-z_1)=0$ where, a, b, and c are the direction ratios of normal.

Here, the direction ratios of normal are 1, 2, and −3 and the point P is (1, 2, −3). Thus, the equation of the required plane is

$$1(x-1)+2(y-2)-3(z+3)=0$$
$$\Rightarrow x+2y-3z-14=0$$

Question 17:

Find the equation of the plane which contains the line of intersection of the planes

$\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k})-4=0$, $\vec{r}\cdot(2\hat{i}+\hat{j}-\hat{k})+5=0$

and which is perpendicular to the plane $\vec{r}\cdot(5\hat{i}+3\hat{j}-6\hat{k})+8=0$.

Answer

The equations of the given planes are

$$\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k})-4=0 \quad ...(1)$$
$$\vec{r}\cdot(2\hat{i}+\hat{j}-\hat{k})+5=0 \quad ...(2)$$

The equation of the plane passing through the line intersection of the plane given in equation (1) and equation (2) is

$$\left[\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k})-4\right]+\lambda\left[\vec{r}\cdot(2\hat{i}+\hat{j}-\hat{k})+5\right]=0$$
$$\vec{r}\cdot\left[(2\lambda+1)\hat{i}+(\lambda+2)\hat{j}+(3-\lambda)\hat{k}\right]+(5\lambda-4)=0 \quad ...(3)$$

The plane in equation (3) is perpendicular to the plane, $\vec{r}\cdot(5\hat{i}+3\hat{j}-6\hat{k})+8=0$

$\therefore 5(2\lambda+1)+3(\lambda+2)-6(3-\lambda)=0$

$\Rightarrow 19\lambda-7=0$

$\Rightarrow \lambda=\frac{7}{19}$

Substituting $\lambda=\frac{7}{19}$ in equation (3), we obtain

$\Rightarrow \vec{r}\cdot\left[\frac{33}{19}\hat{i}+\frac{45}{19}\hat{j}+\frac{50}{19}\hat{k}\right]\frac{-41}{19}=0$

$\Rightarrow \vec{r}\cdot\left(33\hat{i}+45\hat{j}+50\hat{k}\right)-41=0$ ...(4)

This is the vector equation of the required plane.

The Cartesian equation of this plane can be obtained by substituting $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ in equation (3).

$\left(x\hat{i}+y\hat{j}+z\hat{k}\right)\cdot\left(33\hat{i}+45\hat{j}+50\hat{k}\right)-41=0$

$\Rightarrow 33x+45y+50z-41=0$

Question 18:

Find the distance of the point (−1, −5, −10) from the point of intersection of the line $\vec{r}=2\hat{i}-\hat{j}+2\hat{k}+\lambda\left(3\hat{i}+4\hat{j}+2\hat{k}\right)$ and the plane $\vec{r}.\left(\hat{i}-\hat{j}+\hat{k}\right)=5$.

Answer

The equation of the given line is

$\vec{r}\cdot=2\hat{i}-\hat{j}+2\hat{k}+\lambda\left(3\hat{i}+4\hat{j}+2\hat{k}\right)$ ...(1)

The equation of the given plane is

$\vec{r}.\left(\hat{i}-\hat{j}+\hat{k}\right)=5$ ...(2)

Substituting the value of $\vec{r}$ from equation (1) in equation (2), we obtain

$$\left[2\hat{i}-\hat{j}+2\hat{k}+\lambda\left(3\hat{i}+4\hat{j}+2\hat{k}\right)\right]\cdot\left(\hat{i}-\hat{j}+\hat{k}\right)=5$$
$$\Rightarrow\left[(3\lambda+2)\hat{i}+(4\lambda-1)\hat{j}+(2\lambda+2)\hat{k}\right]\cdot\left(\hat{i}-\hat{j}+\hat{k}\right)=5$$
$$\Rightarrow(3\lambda+2)-(4\lambda-1)+(2\lambda+2)=5$$
$$\Rightarrow\lambda=0$$

Substituting this value in equation (1), we obtain the equation of the line as

$$\vec{r}=2\hat{i}-\hat{j}+2\hat{k}$$

This means that the position vector of the point of intersection of the line and the plane is $\vec{r}=2\hat{i}-\hat{j}+2\hat{k}$

This shows that the point of intersection of the given line and plane is given by the coordinates, (2, −1, 2). The point is (−1, −5, −10).

The distance d between the points, (2, −1, 2) and (−1, −5, −10), is

$$d=\sqrt{(-1-2)^2+(-5+1)^2+(-10-2)^2}=\sqrt{9+16+144}=\sqrt{169}=13$$

Question 19:

Find the vector equation of the line passing through (1, 2, 3) and parallel to the planes $\vec{r}=\left(\hat{i}-\hat{j}+2\hat{k}\right)=5$ and $\vec{r}\cdot\left(3\hat{i}+\hat{j}+\hat{k}\right)=6$

.

Answer

Let the required line be parallel to vector $\vec{b}$ given by,

$$\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$$

The position vector of the point (1, 2, 3) is $\vec{a}=\hat{i}+2\hat{j}+3\hat{k}$

The equation of line passing through (1, 2, 3) and parallel to $\vec{b}$ is given by,

$$\vec{r}=\vec{a}+\lambda\vec{b}$$
$$\Rightarrow\vec{r}\left(\hat{i}+2\hat{j}+3\hat{k}\right)+\lambda\left(b_1\hat{i}+b_2\hat{j}+b_3\hat{k}\right)\qquad...(1)$$

The equations of the given planes are

$\vec{r}\cdot\left(\hat{i}-\hat{j}+2\hat{k}\right)=5$ ...(2)

$\vec{r}\cdot\left(3\hat{i}+\hat{j}+\hat{k}\right)=6$ ...(3)

The line in equation (1) and plane in equation (2) are parallel. Therefore, the normal to the plane of equation (2) and the given line are perpendicular.

$\Rightarrow\left(\hat{i}-\hat{j}+2\hat{k}\right)\cdot\lambda\left(b_1\hat{i}+b_2\hat{j}+b_3\hat{k}\right)=0$

$\Rightarrow\lambda\left(b_1-b_2+2b_3\right)=0$

$\Rightarrow b_1-b_2+2b_3=0$ ...(4)

Similarly, $\left(3\hat{i}+\hat{j}+\hat{k}\right)\cdot\lambda\left(b_1\hat{i}+b_2\hat{j}+b_3\hat{k}\right)=0$

$\Rightarrow\lambda\left(3b_1+b_2+b_3\right)=0$

$\Rightarrow 3b_1+b_2+b_3=0$ ...(5)

From equations (4) and (5), we obtain

$$\frac{b_1}{(-1)\times1-1\times2}=\frac{b_2}{2\times3-1\times1}=\frac{b_3}{1\times1-3(-1)}$$

$$\Rightarrow\frac{b_1}{-3}=\frac{b_2}{5}=\frac{b_3}{4}$$

Therefore, the direction ratios of $\vec{b}$ are −3, 5, and 4.

$\therefore\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}=-3\hat{i}+5\hat{j}+4\hat{k}$

Substituting the value of $\vec{b}$ in equation (1), we obtain

$\vec{r}=\left(\hat{i}+2\hat{j}+3\hat{k}\right)+\lambda\left(-3\hat{i}+5\hat{j}+4\hat{k}\right)$

This is the equation of the required line.

Question 20:

Find the vector equation of the line passing through the point (1, 2, − 4) and perpendicular to the two lines: $\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7}$ and $\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$

Answer

Let the required line be parallel to the vector $\vec{b}$ given by, $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$

The position vector of the point (1, 2, − 4) is $\vec{a}=\hat{i}+2\hat{j}-4\hat{k}$

The equation of the line passing through (1, 2, −4) and parallel to vector $\vec{b}$ is

$$\vec{r}=\vec{a}+\lambda\vec{b}$$

$$\Rightarrow \vec{r}\left(\hat{i}+2\hat{j}-4\hat{k}\right)+\lambda\left(b_1\hat{i}+b_2\hat{j}+b_3\hat{k}\right) \qquad ...(1)$$

The equations of the lines are

$$\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7} \qquad ...(2)$$

$$\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5} \qquad ...(3)$$

Line (1) and line (2) are perpendicular to each other.

$$\therefore 3b_1-16b_2+7b_3=0 \qquad ...(4)$$

Also, line (1) and line (3) are perpendicular to each other.

$$\therefore 3b_1+8b_2-5b_3=0 \qquad ...(5)$$

From equations (4) and (5), we obtain

$$\frac{b_1}{(-16)(-5)-8\times7}=\frac{b_2}{7\times3-3(-5)}=\frac{b_3}{3\times8-3(-16)}$$

$$\Rightarrow \frac{b_1}{24}=\frac{b_2}{36}=\frac{b_3}{72}$$

$$\Rightarrow \frac{b_1}{2}=\frac{b_2}{3}=\frac{b_3}{6}$$

∴Direction ratios of $\vec{b}$ are 2, 3, and 6.

$$\therefore \vec{b}=2\hat{i}+3\hat{j}+6\hat{k}$$

Substituting $\vec{b}=2\hat{i}+3\hat{j}+6\hat{k}$ in equation (1), we obtain

$$\vec{r}=\left(\hat{i}+2\hat{j}-4\hat{k}\right)+\lambda\left(2\hat{i}+3\hat{j}+6\hat{k}\right)$$

This is the equation of the required line.

Question 21:

Prove that if a plane has the intercepts a, b, c and is at a distance of P units from the origin, then $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{p^2}$

Answer

The equation of a plane having intercepts a, b, c with x, y, and z axes respectively is given by,

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \qquad ...(1)$$

The distance (p) of the plane from the origin is given by,

$$p=\left|\frac{\frac{0}{a}+\frac{0}{b}+\frac{0}{c}-1}{\sqrt{\left(\frac{1}{a}\right)^2+\left(\frac{1}{b}\right)^2+\left(\frac{1}{c}\right)^2}}\right|$$

$$\Rightarrow p=\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}$$

$$\Rightarrow p^2=\frac{1}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}$$

$$\Rightarrow \frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$$

Question 22:

Distance between the two planes: $2x+3y+4z=4$ and $4x+6y+8z=12$ is

(A)2 units (B)4 units (C)8 units (D) $\frac{2}{\sqrt{29}}$ units

Answer

The equations of the planes are

$$2x+3y+4z=4 \qquad ...(1)$$
$$4x+6y+8z=12$$
$$\Rightarrow 2x+3y+4z=6 \qquad ...(2)$$

It can be seen that the given planes are parallel.

It is known that the distance between two parallel planes, ax + by + cz = d_1 and ax + by + cz = d_2, is given by,

$$D=\left|\frac{d_2-d_1}{\sqrt{a^2+b^2+c^2}}\right|$$
$$\Rightarrow D=\left|\frac{6-4}{\sqrt{(2)^2+(3)^2+(4)^2}}\right|$$
$$D=\frac{2}{\sqrt{29}}$$

Thus, the distance between the lines is $\frac{2}{\sqrt{29}}$ units.

Hence, the correct answer is D.

Question 23:

The planes: 2x − y + 4z = 5 and 5x − 2.5y + 10z = 6 are

(A) Perpendicular (B) Parallel (C) intersect y-axis (D) passes through $\left(0,0,\frac{5}{4}\right)$

Answer

The equations of the planes are

2x − y + 4z = 5 … (1)

5x − 2.5y + 10z = 6 … (2)

It can be seen that,

$$\frac{a_1}{a_2}=\frac{2}{5}$$
$$\frac{b_1}{b_2}=\frac{-1}{-2.5}=\frac{2}{5}$$
$$\frac{c_1}{c_2}=\frac{4}{10}=\frac{2}{5}$$
$$\therefore \quad \frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$$

Therefore, the given planes are parallel.
Hence, the correct answer is B.